

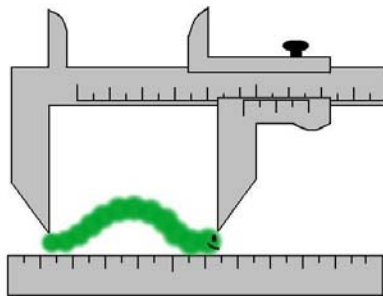
EXPERIMENTAL UNCERTAINTIES

Measurements of any physical quantity can never be exact. One can only know its value with a range of uncertainty. This fact can be expressed in the standard form $X \pm \Delta X$. This expresses the experimenter's judgment that the "true" value of X lies between $X - \Delta X$ and $X + \Delta X$.

Uncertainties of measurements

1. Instrumental uncertainties

Every measuring instrument has an inherent uncertainty that is determined by the *precision of the instrument*. Usually this value is taken as a half of the smallest increment of the instrument scale. For example, 0.5 millimeter is the precision of a ruler; 0.5 sec is the precision of a watch, etc.



Instrumental uncertainties are the easiest ones to estimate, but unfortunately they are not the only source of the uncertainty in your measured value. You must be a skillful and lucky experimentalist to get rid of all other sources and to have the measurement uncertainty equal to the instrumental one.

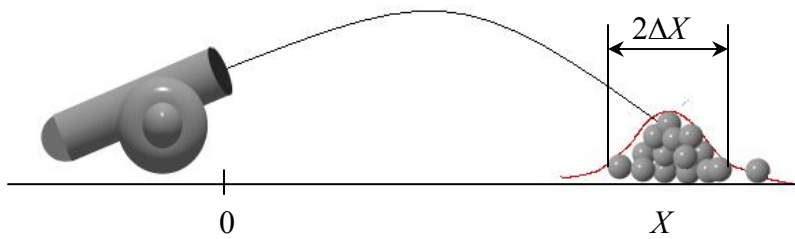
2. Random uncertainties

Sometimes when you measure the same quantity, you can get different results each time you measure it. That happens because different uncontrollable factors affect your results randomly. This type of uncertainty, *random uncertainty*, can be estimated only by repeating the same measurement several times. For example if you measure the distance from a cannon to the place where the cannonball hits the ground, you could get different distances every time you repeat the same experiment. For example, say you took three measurements and obtained 50 m, 51 m, and 49 meters. To estimate the *absolute value of the random uncertainty* you first find the average of your measurements:

$$X = (50\text{m} + 51\text{m} + 49\text{m}) / 3 = 50\text{ m.}$$

You then *estimate approximately* how much the values are spread with respect to this average – in this case we have a spread of about $\Delta X = 1\text{ m}$. That is, our measurement of the distance was

$$X = 50\text{ m} \pm 1\text{ m.}$$

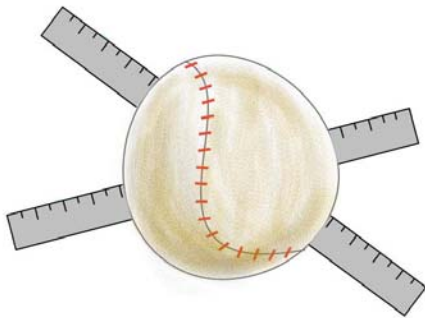


Most of cannonballs will fall in the range from $X - \Delta X$ to $X + \Delta X$.

Thus, multiple trials allow you to find the average value and to estimate the uncertainty range.

3. Effect of assumptions

Assumptions inherent in your model may also contribute in uncertainty. For example, an uneven surface may change the speed of a moving car, or thermal energy loss during a calorimeter experiment may cause the measured temperature to drift. Repeating the measurement will not let you get rid of such effects. This type of uncertainty is not easy to recognize and to evaluate. First of all, you have to determine the sign of the effect, i.e. whether the assumption increases the measured value, decreases it, or affects it randomly. Then you have to try to estimate the size of the effect.



For example, you measure the diameter of the baseball assuming it is a perfect sphere. However, the real size of the ball may differ by 1 or 2 mm if you measure in different dimensions. This difference will determine the uncertainty of your measurement.

It is difficult to give strict rules and instructions on how to estimate uncertainties in general. Each case is unique and requires a thoughtful approach. Be ingenious and reasonable.

Comparing uncertainties

By looking at the absolute uncertainty for two quantities (ΔX and ΔY) you cannot immediately decide which quantity is more uncertain. This is because the units and the magnitudes of the measured quantities are probably different. How can we decide which quantity has a larger uncertainty? We need to compare their *relative uncertainties*, the ratio of the *absolute* uncertainty and the quantity itself $\Delta X/X$. This may be expressed as a fraction or as a percentage by multiplying the ratio by 100%.



The two circles in the picture have similar fuzzy edges but the larger circle looks sharper. Why? The diameter D of the larger circle is about 90 units (arbitrary units) and on the same scale the smaller circle has a diameter of about 30-units. The *absolute uncertainty* ΔD is the same for each circle, about 9 units. However, the *relative uncertainty* $\Delta D/D$ is about 10% for the large circle and about 30% for the small one. The relative uncertainty is a better indicator of the uncertainty of the measured quantity.

Note: Common sense and good judgment must be used in representing the uncertainties when stating a result. Consider a temperature measurement with a thermometer known to be reliable to ± 0.5 degree Celsius. Would it make sense to say that this causes a 0.5% uncertainty in measuring the boiling point of water (100 degrees) but a whopping 10% in the measurement of cold water at a temperature of 5 degrees? Of course not! (And what if the temperatures were expressed in degrees Kelvin? That would seem to reduce the relative uncertainty to insignificance!). However in most calorimetry tasks, the value of interest is not temperature itself but only the change of the temperature or the temperature difference.

Reducing uncertainties

The example with the circles shows a way to reduce relative uncertainty in a measurement. The same absolute uncertainty yields a smaller relative uncertainty if the measured value is larger. Suppose you have a block attached to a spring and want to measure the time interval for it to oscillate up and down, back to its starting position. If you use a watch to measure the time interval, the absolute uncertainty of the measurement is about 0.5 s. If you now measure a single time interval of 5 s, you get a relative uncertainty of 10% $[(0.5 \text{ s}/5 \text{ s})100]$. Suppose you measure the time interval for 5 oscillations instead and you measure 25 s. The instrumental uncertainty is *still* 0.5 s! The *relative uncertainty* in your measurement of the time interval is now:

$$\text{time interval relative uncertainty} = (0.5 \text{ s}/25 \text{ s}) * 100\% = 2\%$$

By measuring a longer time interval (five oscillations instead of one), you have reduced the uncertainty in your time interval measurement by a factor of 5!

Of course you should not forget about the obvious way of reducing relative uncertainties by minimizing absolute uncertainty with a better design, decreasing the effect of assumptions, or increasing the accuracy of instrument if it is possible.

Uncertainty in the final calculated value

Suppose you want to determine the uncertainty in the final value of a quantity that is calculated from several measured quantities. The uncertainties in these measured quantities propagate through the calculation to produce uncertainty in the final result. Consider the following example.

Suppose you know the average mass of one apple m with the uncertainty Δm . If you want to calculate the mass of the basket of 100 apples, you will get the value $M \pm \Delta M = 100 m \pm 100 \Delta m$. The *relative uncertainty* of calculated value of M remains the same as the relative uncertainty of the single measured value for m

$$\Delta M / M = \Delta m / m.$$

If you have more than one measured quantity, estimating uncertainty may be more complicated. However, if one of your sources of relative uncertainty is *much larger* than the others, then you can neglect other sources and use the *weakest link rule*.

Weakest link rule

The percent uncertainty in the calculated value of some quantity is at least as great as the greatest percentage uncertainty of the values used to make calculation. Thus to estimate uncertainty in you calculated value, you have to:

1. Estimate the *absolute uncertainty* in each measured quantity used to find the calculated quantity.
2. Calculate the *relative percentage uncertainty* in each measured quantity.
3. Pick the largest *relative percentage uncertainty*. This is the weakest link.
4. Apply the relative uncertainty of the weakest link to the calculated quantity to determine its absolute uncertainty.

Here's an example: You've been asked to estimate the volume of your laptop computer. First, you measure the length, width, and thickness with a meter stick (which has an absolute uncertainty of 0.05cm).

Measurement	Value (with absolute uncertainty)	Relative uncertainty
Length	$39.4 \pm 0.05\text{cm}$	$\frac{0.05\text{cm}}{39.4\text{cm}} = 1.27 \times 10^{-3} = 0.127\%$
Width	$28.7 \pm 0.05\text{cm}$	$\frac{0.05\text{cm}}{28.7\text{cm}} = 1.74 \times 10^{-3} = 0.174\%$
Thickness	$4.3 \pm 0.05\text{cm}$	$\frac{0.05\text{cm}}{4.29\text{cm}} = 11.6 \times 10^{-3} = 1.16\%$

The thickness has by far the largest relative uncertainty. The volume of the laptop is

$$V = LWT = (39.4\text{cm})(28.7\text{cm})(4.3\text{cm}) = 4862\text{cm}^3.$$

To determine the absolute uncertainty multiply the volume by the relative uncertainty of the weakest link

$$\Delta V = (4860\text{cm}^3)(11.6 \times 10^{-3}) = 56\text{cm}^3$$

So, the final estimate for the volume of the laptop is

$$V = 4862 \pm 56\text{cm}^3$$

Comparable uncertainties

If a final calculated value depends on several measured quantities that each has comparable uncertainties, then the rules are more complicated: they depend on the type of the mathematical relationship you use for the calculation. In the most frequent case of multiplying two quantities together, their relative uncertainties add. If a final calculated value depends on the second power of a measured quantity (for example, the area of a circle depends on the second power of its measured radius), then the relative uncertainty of the

calculated quantity is twice the relative uncertainty of the measured quantity. If the calculated value depends on the third power of a measured quantity, then the relative uncertainty in the calculated value is three times the relative uncertainty of the measured quantity. For example, the relative uncertainty of the volume of a baseball will be three times larger than the relative uncertainty in its measured radius.

Why do you need to know uncertainty?

Is the measured value in agreement with the prediction? Does the data fit the physical model? Are two measured values the same? You cannot answer these questions without considering the uncertainties of your measurements. Indeed, are the values of two quantities the same if the difference between them is smaller than the uncertainty in their measurements?



Which bunch of grass is higher? You cannot tell this because the difference in their average heights is smaller than the uncertainties in their heights. If you cannot tell which of two values is larger, you can claim that they are the same.

Thus, to make judgment about two values X and Y , you have to find the ranges where these values lie. If the ranges $X \pm \Delta X$ and $Y \pm \Delta Y$ overlap, you can claim that the values X and Y are the same within your experimental uncertainty.

Summary

When you are doing a lab and measuring some quantities to determine an unknown quantity:

- Decide which factors affect your result the most.
- Wherever possible, try to reduce the effects of these factors that cause uncertainty.
- Wherever possible, try to reduce uncertainties by measuring longer distances or time intervals, etc.
- Decide the absolute uncertainty of each measurement.
- Then, find the relative uncertainty of each measurement.
- If one relative uncertainty is much larger than the others, you can ignore all other sources and use this uncertainty to write the value of the relative uncertainty of the quantity that you are calculating.
- Find the range where your calculated quantity lies. Make a judgment about your results taking into the account this uncertainty.

Exercises (Solutions below so you can check your work)

1. What is the percent uncertainty in the length measurement 2.35 ± 0.25 m?
2. Suppose that after a hike in the mountains, a friend asks how fast you walked. You recall that the trail was about 6 miles long and that it took between two and three hours. What is your average speed if you assume the average time of 2.5 h?
Assume that absolute uncertainty in your distance measurement is 0.1 mile. Estimate the relative (percentage) uncertainty in your distance measurement?
What is an absolute uncertainty in your time measurement? What is its relative uncertainty?
Compare the relative uncertainties in time and distance. Which measurements are more accurate? Determine whether you can use the weakest link rule. Determine the relative and absolute uncertainties in the speed estimation.
3. Suppose you want to measure time for the ball falling from a height of 1 m. You took three measurements of the time interval and obtained 0.5 s, 0.6 s, and 0.4 seconds. What is average time of the fall? What is an absolute value of the random uncertainty in the time measurement? What is the relative uncertainty?
4. Suppose now that you measured the time interval for the ball to fall from a 10-m height and got 1.5s, 1.7s, and 1.6s. Estimate the relative uncertainty assuming the absolute value is the same as in the previous task. Compare the relative uncertainties in tasks 2 and 3. Make a conclusion.
5. You drive along highway and want to estimate your average speed. You notice a sign indicating that it is 260 miles to Boston. In 45 min you pass another sign indicating 210 miles to Boston. Make reasonable assumptions for the absolute uncertainties of your time and distance measurements. Estimate relative (percentage) uncertainties. State your average speed with the uncertainty.
6. You want to know how fast your coffee is cooling in your mug. For this you measure temperature with a thermometer. Your first measurement is 76 ± 1 °C (you use the usual thermometer with the smallest increment 1°C). In 15 min temperature is 68 ± 1 °C. What is the temperature drop (state the uncertainty range)? What is the relative uncertainty in your measurement?

Solutions

1. The percent uncertainty, also called 'relative uncertainty' is

$$\frac{0.25m}{2.35m} = 0.106 = 10.6\%$$

2. Your average speed is

$$\frac{6mi}{2.5h} = 2.4mph$$

The relative uncertainty in the distance measurement is

$$\frac{0.1mi}{6mi} = 0.0167 = 1.67\%$$

The absolute uncertainty in the time measurement is 0.5h since the hike might have taken as short as 2h or as long as 3h. The relative uncertainty in the time measurement is

$$\frac{0.5h}{2.5h} = 0.2 = 20\%$$

The distance measurement is more accurate because its relative uncertainty is smaller. Because it is significantly smaller the weakest link may be used to estimate the uncertainty in the speed. The weakest link rule says to use the largest relative uncertainty from the data as the relative uncertainty of anything you calculate from the data. So, the relative uncertainty in the speed is 20%. The absolute uncertainty in the speed is then

$$2.4mph \times 0.2 = 0.48mph$$

3. The average time of the fall is

$$\frac{0.5s + 0.6s + 0.4s}{3} = 0.5s$$

The absolute uncertainty is 0.1s since all the trials lie within 0.1s of the average. The relative uncertainty is

$$\frac{0.1s}{0.5s} = 0.2 = 20\%$$

4. The average is 1.6s and the relative uncertainty is

$$\frac{0.1s}{1.6s} = 0.0625 = 6.25\%$$

The measurement of the time of fall from 10m is more accurate than the time of fall from 1m since the relative uncertainty is lower.

5. The distance measurements have an absolute uncertainty of 0.5mi. The 210mi measurement has the larger relative uncertainty since it is the smaller value. It is equal to

$$\frac{0.5mi}{210mi} = 0.00238 = 0.238\%$$

However, this relative uncertainty is going to be comparable to the relative uncertainty of the 260mi measurement. Normally this would mean I should find the relative uncertainty of both and add them. But, the weakest link is going to be the time measurement anyway, so I don't have to bother. The absolute uncertainty in the time measurement is 0.5min. Its relative uncertainty is

$$\frac{0.5\text{min}}{45\text{min}} = 0.0111 = 1.11\%$$

The average speed is

$$\frac{260\text{mi} - 210\text{mi}}{45\text{min} \left(\frac{1\text{h}}{60\text{min}} \right)} = 66.7\text{mph}$$

Using the weakest link rule, the relative uncertain in the speed is 1.11%, so the absolute uncertainty is $66.7\text{mph} \times 0.0111 = 0.7\text{mph}$. The average speed is then

$$66.7 \pm 0.7\text{mph}$$

6. The temperature drop is $76^\circ\text{C} - 68^\circ\text{C} = 8.0^\circ\text{C}$. To find the absolute uncertainty in this I need to determine the relative uncertainties of both temperature measurements. They are

$$\frac{1^\circ\text{C}}{76^\circ\text{C}} = 0.0132 = 1.32\%$$

$$\frac{1^\circ\text{C}}{68^\circ\text{C}} = 0.0147 = 1.47\%$$

Since these are comparable I have to add the relative uncertainties and use that as the relative uncertainty for the temperature drop. This makes the absolute uncertainty of the temperature drop $8.0^\circ\text{C} \times (0.0132 + 0.0147) = 0.2$. So, the temperature drop is

$$8.0 \pm 0.2^\circ\text{C}$$
