# 1 Introduction

# 2 Statement of results

## 2.1 Some preliminary definitions

A **strict partition** $\lambda$ is a strictly decreasing sequence of positive integers. For convenience, we extend the definition of a partition to an infinite sequence, with the strictly decreasing sequence followed by an infinite string of trailing 0's. The **length** $l(\lambda)$ of $\lambda$ is the numbers of "parts," i.e. nonzero elements of the sequence. We denote the set of strict partitions with all parts at most $k$ by $SP(k)$, for $k > 0$ an integer. The **weight** of $\lambda$, denoted by $|\lambda|$, is the sum of all of the elements of the sequence.

The **shifted diagram** $\mathrm{Sd}(\lambda)$ of $\lambda$ is a set of "boxes" arranged on a grid. In the $i$th row, there are $\lambda_i$ boxes, and they begin at the $i$th column and end at the $i + \lambda_i$th column. If $B$ is a box in a shifted diagram, $co(B)$ is the column $B$ is in, and $r(B)$ is the row $B$ is in. $c(B)$ will refer to the quantity $co(B) - r(B)$, which is usually referred to as the content of the box. Some shifted diagrams are shown in the figure. $l(B)$ is the **leg length** of $B$, which is the number of boxes directly below $B$ in the shifted diagram, i.e. $\#\{i > r(B) : i \leq co(B) \leq \lambda_i + i - 1\}$, and $a(B)$ refers to the **arm length** of $B$, which is the number of boxes directly to the right of $B$ in the shifted diagram, or in other words $\lambda_{r(B)} - co(B) + i - 1$.

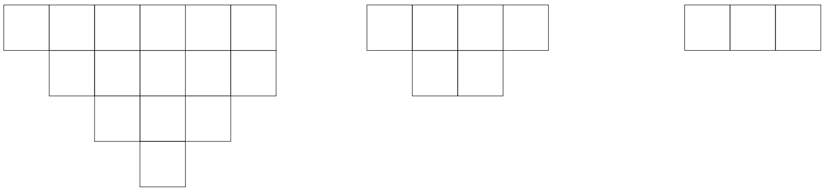


Figure 1: The shifted diagrams of (6531), (42), and (3).

**Definition 2.1.** For a box $B$ in the shifted diagram of a strict partition $\lambda$, the **diagonal length** of $B$ is the number of complete boxes the line that starts at the center of $B$ and continues diagonally to the right and downward passes through, which is equal to $\#\{i > r(B) : \lambda_i > c(B)\}$. Equivalently, this is the standard leg length of $B$ in the unshifted diagram of $\lambda$.

**Definition 2.2.** A box $B$ in a shifted diagram is said to be an **outside corner** if there is no box to the right of $B$ and no box below $B$. In the unshifted diagram, this translates to there being no box to the right of $B$ and no box diagonally to the left and downward of $B$.

A **barred shifted tableau** $T$ of shape $\lambda$ is a certain type of labelling of the boxes in the shifted diagram of $\lambda$, in this case with numbers, some of which have bars over them. The numbers must be strictly increasing down the rows, and weakly increasing across the columns from left to right, without regard to bars. $|B|$ will be the number inside $B$. $B(T)$ will refer to the set of all boxes $B$ in $T$ such that $B$ is labelled with a barred number.

A barred shifted tableau $T$ is said to be of **type** $D$ if for all $B \in T$ with $c(B) = 0$, $|B|$ is even if and only if $r(B)$ is even.

We fix a total ordering $<$ on ordered pairs of positive integers in an $n \times n$ box, so that we have an ordering of the boxes in a shifted diagram (and hence in a tableau). The particular ordering chosen will be shown to be irrelevant.

**Definition 2.3.** For a barred shifted tableau $T$, the set $B(T)$ will be indexed by positive integers according to its ordering under $<$. For convenience in notation, we define the $i$th "block" in $T$, $\mathrm{Bl}_T^i$, by

$\mathrm{Bl}_T^1 = \{B \in T : B < B(T)_1\}$,
$\mathrm{Bl}_T^i = \{B \in T : B(T)_i < B < B(T)_{i+1}\}$
for $1 < i < |B(T)|$, and
$\mathrm{Bl}_T^{|B(T)|} = \{B \in T : B > D \text{ for all } D \in B(T)\}$.

**Definition 2.4.** If $\mu$ is a strict partition, a barred shifted tableau $T$ is said to be **compatible** with $\mu$ if for all $B \in T$, $\mu_{|B|} > c(B)$. We denote by $\overline{SSYT}(\lambda, \mu, k)$ the set of all barred shifted tableaux of shape $\lambda$ that are compatible with $\mu$, in which $k$ boxes are barred. To denote the set of shifted tableaux of shape $\lambda$ without bars that are compatible with $\mu$, we will write $SSYT(\lambda, \mu)$.

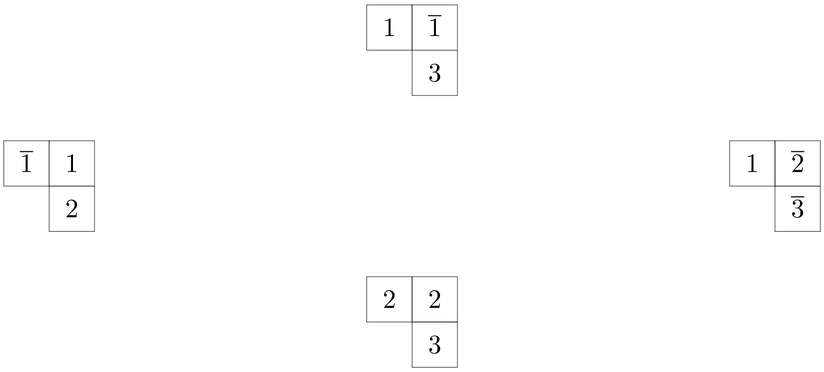


Figure 2: All shifted tableaux of shape (21) compatible with (321). The left-hand one is of type $D$. Bars can be placed arbitrarily.

**Definition 2.5.** For a box $B$ in a shifted tableau, we denote its **weight** with respect to a strict partition $\lambda$ by $B|_\lambda \in \mathbb{Z}[t_1, \ldots, t_n]$. We postpone here the definition of the weight, because it depends on the choice of $G$ and, in the case where $G = SO(2n, \mathbb{C})$, on the choice of parabolic subgroup $P \subset G$. See Definitions 3.3 (type $C$), 3.5 (type $B$), and 3.7 (type $D$).

If $B$ is not a member of a shifted tableau compatible with $\lambda$, then $B|_\lambda = 0$. It will be shown later that $B|_\lambda$ is either a positive root with respect to a basis of simple roots $(\alpha_1, \ldots, \alpha_n)$, or equal to 0, for all $B$ and $\lambda$.

We also define the weight of a block to be

$$\mathrm{Bl}_T^i|_\lambda = \prod_{B \in \mathrm{Bl}_T^i} B|_\lambda.$$

**Definition 2.6.** A **path** $P$ of length $k$ for $k \geq 0$ is a sequence of strict partitions $(P^{(0)}, P^{(1)}, \ldots, P^{(k)})$, with $P^{(i-1)} \subset P^{(i)}$, such that the **skew diagram** $\mathrm{Sd}(P^{(i)})/\mathrm{Sd}(P^{(i-1)})$ (the set of all boxes in $P^{(i)}$ that are not in $P^{(i-1)}$) contains exactly one box for all $0 < i \leq n$. $P$ is said to be "from" $P^{(0)}$, and "to" $P^{(k)}$. The notation $\mathrm{Paths}(\lambda, \nu)$ will be used to refer to the set of all paths from $\lambda$ to $\nu$.

A word of warning: if $\lambda$ is a partition, we occasionally use $\lambda$ in place of $\mathrm{Sd}(\lambda)$. The meaning should be clear from the context.

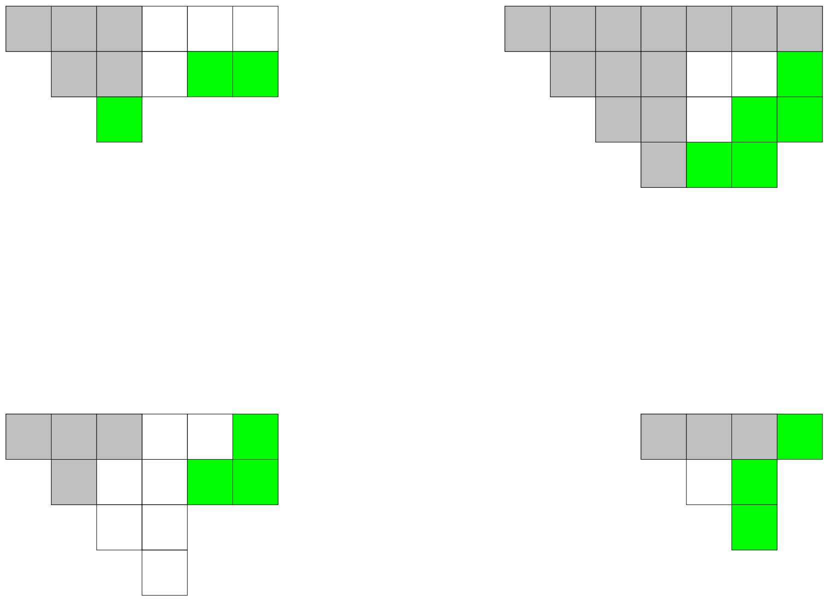


Figure 3: Some triples of strict partitions satisfying the outside corner condition.

## 2.2 The Positive Formula

We later define a positive integer-valued function $r_i(B, \lambda, \eta)$ ,where $B$ is a box in a (barred) shifted tableaux, and $\lambda$ and $\eta$ are strict partitions. As with the weight, the definition depends on the choice of $G$ and $P$. See Definitions 3.4 (type $C$), 3.6 (type $B$), and 3.8 (type $D$).

We define
$r(T, P) := \prod_{i=1}^{l(P)} r_{1+c(P^{(i)}/P^{(i-1)})}(B(T)_i, P^{(i-1)}, P^{(l(P))})$,
where $T$ is a barred shifted tableau, and $P$ is a path. Also, we define

$$w(T, P) = \prod_{i=1}^{l(P)} \mathrm{Bl}_T^i|_{P^{(i)}}$$

**Definition 2.7.** A triple $(\mu, \lambda, \nu)$ of strict partitions is said to satisfy the **outisde corner condition** if for each outside corner $B$ in the shifted diagram of $\mu$, the diagonal length of $B$ when the shifted diagram of $\mu$ is imbedded into the shifted diagram of $\lambda$ is the same as the diagonal length of $B$ when the shifted diagram of $\mu$ is imbedded into the shifted diagram of $\nu$.

We show

**Theorem 1.** *Let $(\mu, \lambda, \nu)$ be a triple of strict partitions satisfying the outside corner condition (Definition 2.7). Then the formula*

$$c_{\lambda\mu}^{\nu} = \sum_{\substack{T \in \overline{SSYT}(\mu,\nu,k) \\ P \in Paths(\lambda,\nu)}} r(T, P) w(T, P)$$

*holds for types $B$, $C$, and $D$. In particular, $c_{\lambda\mu}^{\nu}$ has nonnegative integer coefficients when expressed in terms of the simple roots.*

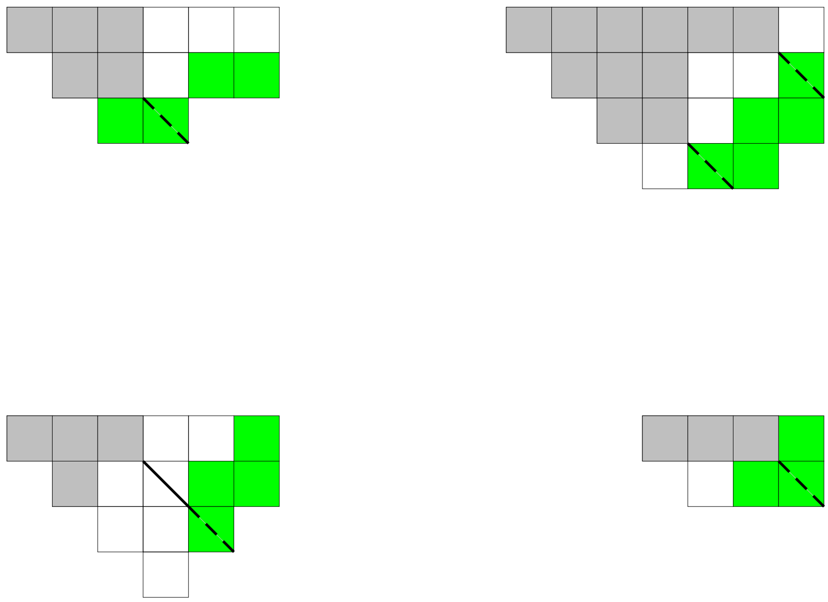


Figure 4: Some triples of strict partitions that do NOT satisfy the outside corner condition.

## 2.3 Restrictions of Schubert classes to fixed points

The ring $H_T^*(G/P)$ can be imbedded into a direct product of polynomial rings, as follows:

The action of $T$ on $G/P$ has $2^n$ fixed points, one for each Schubert cell. We denote these fixed points by $x_\lambda$ for $\lambda$ a strict partition. The inclusion map $x_\lambda \hookrightarrow G/P$ induces a restriction homomorphism $H_T^*(G/P) \to H_T^*(x_\lambda)$. Thus we have a map

$$H_T^*(G/P) \to \prod_\lambda H_T^*(x_\lambda),$$

which happens to be injective.

For the rest of the exposition, we will suppress the $T$ in the notation $[X_\lambda]^T$ for the equivariant Schubert class. We denote the restriction of the Schubert class $[X_\lambda]$ to the fixed point $\mu$ by $[X_\lambda]|_\mu$. We later derive a formula for these restrictions in terms of shifted tableaux:

**Proposition 2.1.** *Let $\lambda, \mu$ be strict partitions. Then*

$$[X_\lambda]|_\mu = \sum_{T \in SSYT(\lambda,\mu)} \prod_{B \in T} B|_\mu,$$

*where the weight $B|_\mu$ of a box is defined according to the choice of Grassmannian.*

This is a new formula, but not the only formula; see, for example, [IN].

As a consequence of the proof, we will see that

**Corollary 2.1.** *If $\lambda$ is a partition, and $B \in T$ is a box in a shifted tableau, then $B|_\lambda$ is either 0 or a positive root with respect to the basis $(\alpha_1, \ldots, \alpha_n)$ of simple roots for a root system of type $B_n$, $C_n$, or $D_n$, depending on the choice of Grassmannian.*

We will need the following proposition.

**Proposition 2.2.** *Let $\lambda$, $\mu$, and $\nu$ be strict partitions. Then*

$$[X_\nu]|_\nu c^\nu_{\lambda\mu} = [X_\lambda]|_\nu([X_\mu]|_\nu - [X_\mu]|_\lambda) - \sum_{\substack{\eta\neq\lambda\\ \eta\subsetneq\nu}} c^\eta_{\lambda\mu}[X_\eta]|_\nu$$

*Proof:* Recall that

$$[X_\lambda]\cdot[X_\mu] = \sum_\eta c^\eta_{\lambda\mu}[X_\eta].$$

If we take

$$X = [X_\lambda][X_\mu] - \sum_{\eta\neq\nu} c^\eta_{\lambda\mu}[X_\eta],$$

then

$$X = c^\nu_{\lambda\mu}[X_\nu].$$

Therefore,

$$X|_\nu = c^\nu_{\lambda\mu}[X_\nu]|_\nu,$$

and the result follows. ■

# 3 The definition of the weights

## 3.1 Preliminary combinatorics

**Definition 3.1.** Let $\lambda$, $\eta$ be strict partitions. A **compatible imbedding** (of shifted diagrams) is a function $f : \mathrm{Sd}(\lambda) \to \mathrm{Sd}(\eta)$ satisfying:

(1) $f$ is injective.

(2) $c(f(B)) = c(B)$ for all $B \in \mathrm{Sd}(\lambda)$.

(3) If $B, C \in \mathrm{Sd}(\lambda)$ and $r(B) > r(C)$, then $r(f(B)) > r(f(C))$.

(4) If $B, C \in \mathrm{Sd}(\lambda)$, $r(B) = r(C)$, and $c(B) < c(C)$, then $r(f(B)) \le r(f(C))$.

$f$ is said to be of **type** $D$ if, in addition,

(5) If $B \in \mathrm{Sd}(\lambda)$ and $c(B) = 0$, then $r(f(B))$ is even if $r(B)$ is even, and odd if $r(B)$ is odd.

The reason for the word "compatible" is clarified in the following lemma.

**Lemma 3.1.** *Let $\lambda$, $\mu$ be strict partitions. The set of compatible imbeddings $Sd(\mu) \to Sd(\lambda)$ is in bijective correspondence with the set of semistandard shifted tableaux of shape $\mu$ compatible with $\lambda$ in such a way that a tableau is of type $D$ if and only if the corresponding imbedding is of type $D$.*

*Proof:* Let $T \in SSYT(\mu, \lambda)$, and define $f_T : \mathrm{Sd}(\mu) \to \mathrm{Sd}(\lambda)$ by setting $r(f_T(B)) = |B|$ and $c(f_T(B)) = c(B)$. We assert that $f_T$ is a compatible imbedding. Condition (2) is satisfied trivially. (3) is satisfied by the fact that if $r(C) > r(B)$, then $|C| > |B|$, since $T$ is a tableaux. (4) is satisfied by the fact that if $r(B) = r(C)$ and $c(B) < c(C)$, then $|B| \le |C|$. What remains to be shown is injectivity.

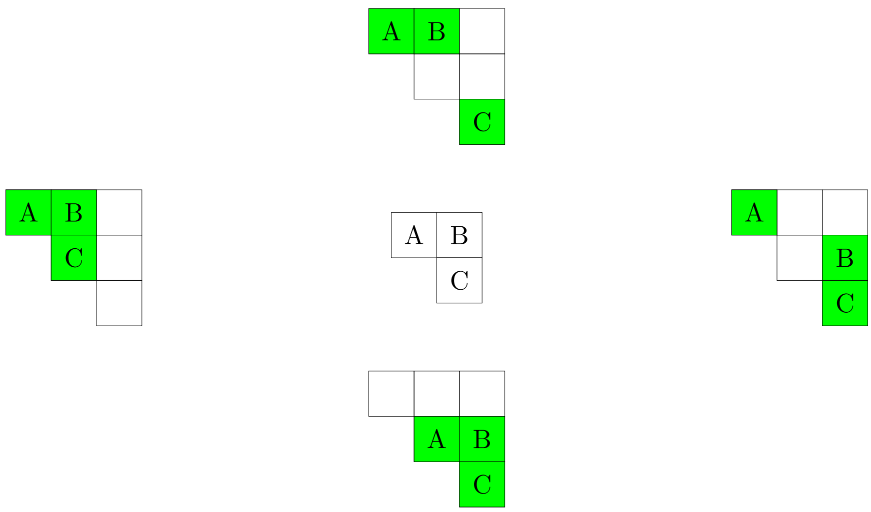
Suppose $f_T(B) = f_T(C)$ for some $B, C \in T$. Then $c(B) = c(C)$. If $B \neq C$, then we can assume $r(B) < r(C)$. In that case, we would have that $|B| < |C|$, because $|B| < |C|$ if $c(B) \le c(C)$ and $r(B) < r(C)$. But we know that $|B| = |C|$ by the definition of $f$, so $r(B) = r(C)$. Thus $B = C$, and $f_T$ is injective. From this construction, it is clear that $f_T$ is of type $D$ if $T$ is.

Let $imb : SSYT(\mu, \lambda) \to \{\text{compatible imbeddings } \mathrm{Sd}(\mu) \to \mathrm{Sd}(\lambda)\}$ such that $imb(T) = f_T$. We show that $imb$ is bijective by constructing its inverse. For a compatible imbedding $f : \mathrm{Sd}(\mu) \to \mathrm{Sd}(\lambda)$, define a tableaux $tab(f)$ by setting, for $B \in \mathrm{Sd}(\mu)$, $|B| = r(f(B))$. $tab(f)$ is clearly a tableaux by conditions (3) and (4) in the definition of compatible imbedding. For $T \in SSYT(\mu, \lambda)$, $tab(imb(T))$ assigns to $B \in T$ the number $r(imb(T)(B)) = |B|$, so $tab \circ imb$ is the identity. Similarly, $imb \circ tab$ is the identity, so the lemma follows. ■

With this in mind, we make the following definitions:

**Definition 3.2.** Let $\mu$, $\lambda$ be strict partitions, and let $T$ be a shifted tableaux of shape $\mu$ compatible with $\lambda$. For a box $B \in T$, We denote by $l_\lambda(B)$ the quantity $l(f(B))$, where $f$ is the compatible imbedding corresponding to $T$. Similarly, $a_\lambda(B) := a(f(B))$. These definitions make sense for barred tableaux as well.

Figure 5: The compatible imbeddings corresponding to the shifted tableaux in Figure 2.




## 3.2 The Lagrangian Grassmannian

This is the choice $G = Sp(2n, \mathbb{C})$. There is essentially only one choice of parabolic subgroup $P \subset G$, and $G/P = LGr(n, 2n)$.

**Definition 3.3.** The formula for the weight of a box in type $C$ is:

$$B|_\lambda = -t_{n+1-\lambda_{|B|}} - t_{n+1-\lambda_{|B|+c(B)}+c(B)-l_\lambda(B)},$$

where $t_k := -t_{2n+1-k}$ for $k > n$.

**Definition 3.4.** The function $r_i$ mentioned previously is defined to be

$$\begin{aligned} r_i(B, \lambda, \eta) := {} & \delta_{n+1-i, n+1-\eta_{|B|}}(\eta_{|B|} - \lambda_{|B|}) \\ & + \delta_{n+1-i, n-c(B)+l_\eta(B)}(l_\eta(B) - l_\lambda(B)) \\ & + \delta_{n+1-i, n+1-\eta_{|B|+c(B)}}\delta_{l_\lambda(B), c(B)}(\eta_{|B|+c(B)} - \lambda_{|B|+c(B)}) \end{aligned}$$

in type $C$.

The root system of $Sp(2n, \mathbb{C})$ is of type $C_n$, and the simple roots we choose are:

$$\alpha_i = t_{i+1} - t_i$$

for $i < n$, and

$$\alpha_n = -2t_n.$$

We recall the action of the Weyl group $W$ of our root system of type $C_n$ on the simple roots. Consider $t_1, \dots, t_n$ to be an orthonormal basis of a real vector space with inner product $(-,-)$. The Weyl group of type $C_n$ is generated by the simple reflections $\sigma_i$ for $1 \le i \le n$, defined by

$$\sigma_i v = v - 2\frac{(\alpha_i, v)}{(\alpha_i, \alpha_i)}\alpha_i.$$

The action can be summarized as follows:

$$\sigma_i \alpha_j = \begin{cases} -\alpha_j & \text{if } i = j \\ \alpha_j & \text{if } |i-j| > 1 \\ \alpha_i + \alpha_j & \text{if } j < n \text{ and } |i-j| = 1 \\ 2\alpha_{n-1} + \alpha_n & \text{if } j = n \text{ and } i = n-1. \end{cases}$$

Let $W_P \subset W$ be the parabolic subgroup generated by $\sigma_n$. Then the left cosets $wW_P$ for $w \in W$ correspond bijectively to strict partitions with all parts less than or equal to $n$. In each of these cosets, there is a unique shortest element; these are used to index the Schubert classes.

$W$ can be thought of as the group of signed permutations on a set with $n$ elements. With our choice of root system, it is convenient to write the set as $(n \cdots 321)$, and the action of the group is on the right. $\sigma_i$ for $1 \le i < n$ permutes the $i$th and $i+1$st entries, and $\sigma_n$ inverts the sign of the last entry. Thus, if $n = 4$, $\sigma_3\sigma_4\sigma_3\sigma_1$ would produce $(34\bar{2}1)$, where the bar indicates negation.

It turns out that the minimal-length elements of the $W_P$-cosets correspond precisely to those permutations that produce decreasing sequences. The partition can be read off as the trailing negative entries. To prove the weight formula, we examine the combinatorics of the Weyl group.

Let $w \in W$. An expression for $w$ in terms of a product of generators is said to be **reduced** if the number of generators in the expression is minimal. The number of generators in such an expression is denoted by $l(w)$. To each finite sequence of positive integers less than or equal to $n$, there corresponds an expression for some Weyl group element: if $(s_1, s_2, \dots, s_k)$ is the sequence, then the corresponding expression is $\sigma_{s_1} \dots \sigma_{s_k}$. Such a sequence is called a **word** when considered in this way. If the expression corresponding to a word is reduced, then the word is said to be a **reduced word**.

A **reverse lattice word** is a (finite) sequence $(a_i)$ of positive integers such that if one were to read the sequence from the right and stop at any point, there would be at least as many $n$'s as $n-1$'s, at least as many $n-1$'s as $n-2$'s, etc. This agrees with usual definition of a reverse lattice word if we replace $i$ with $n+1-i$.

**Lemma 3.2.** *Let $u \in W$, and let $w \in uW_P$ be the unique shortest element. If $b = b_1 \cdots b_{l(w)}$ is a reduced word for $w$, then $b$ is a reverse lattice word. Furthermore, if $c$ is another reduced word for $w$, then for each $k$, $k$ occurs exactly as many times in $b$ as in $c$.*

*Proof:* We prove something stronger: for any minimal length representative $v$ of a $W_P$-coset, every reduced expression for $v$ can be obtained by permuting commuting generators in a reduced expression satisfying that if $j < n$, then $\sigma_j$ is followed by a $\sigma_{j+1}$. This is obvious for $v = \sigma_n$. Assume the result holds for all $v$ with $l(v) < k$, and let $v' = \sigma_j v$, with a reduced expression for $v'$ given by $\sigma_j$ followed by a reduced expression for $v$.

If $j = n$, then there is no problem. Otherwise, move $\sigma_j$ past every generator it commutes with; this can be done until it hits a $\sigma_{j+1}$ or a $\sigma_{j-1}$. If the $\sigma_{j+1}$ is first, then we have the desired reduced expression for $v'$. Otherwise, we can assume we have $\sigma_j\sigma_{j-1}\sigma_j$ in the expression, by induction. This is equal to $\sigma_{j-1}\sigma_j\sigma_{j-1}$. Since $v'$ is a minimal-length coset representative by assumption, we can read the expression from the right and terminate at the last $\sigma_j$, to obtain another minimal-length coset representative $v''$ (for a different coset) that has a reduced expression beginning with $\sigma_j\sigma_{j-1}$. Clearly the $\sigma_j$ cannot be moved past the $\sigma_{j-1}$ using commutation relations (since $\sigma_{j-1}\sigma_j \ne \sigma_j\sigma_{j-1}$), so the last $\sigma_j$ cannot be followed by a $\sigma_{j+1}$. Since $l(v'') < k$, this is a contradiction, by induction.

This proves the "furthermore"; to see that $b$ is a reverse lattice word, note that it can be obtained from a reduced word $d$ consisting of increasing sequences terminated by $n$'s by permuting the numbers according to commutation relations. Since $d$ is a reverse lattice word, and, for any $k$, a $k+1$ cannot be moved past a $k$ permuting as such, we see that $b$ is a reverse lattice word as well. ■

**Proposition 3.1.** *Let $\lambda, \mu$ be strict partitions, with corresponding minimal length $W_P$-coset representatives $v, w$ respectively. Choose a reduced word $b$ for $w$. Then the set of subsequences of $b$ that are reduced words for $v$ is in bijective correspondence with the set $SSYT(\lambda, \mu)$.*

*Proof:* Let $p_d : \mathrm{Sd}(\mu) \to [|\mu|]$ for $d$ a reduced word of a minimal-length coset representative be defined by $p_d(B) = i$, where $i$ is the $r(B)$th index such that $d_i = n - c(B)$, when reading from right to left. Let $g_d : \mathscr{P}([|\mu|]) \to W$ for a reduced word $d$ be defined by $g_d(S) = \sigma_{d_{S_1}} \sigma_{d_{S_2}} \cdots \sigma_{d_{S_m}}$, where $m = |S|$, and $S_1 < S_2 < \cdots < S_m$.

At this point, it is not at all clear that $p_d$ is a bijection. Let $d$ be the word $d^{(l(\mu))} d^{(l(\mu)-1)} \ldots d^{(1)}$, where $d^{(i)} = (n+1-\mu_i, n+2-\mu_i, \ldots, n)$. $d$ is a reduced word for $w$, and some checking shows that $p_d$ is bijective. It then follows from Lemma 3.2 that the function is bijective for any reduced word for $w$.

We wish to show that, for any reduced words $c, d$ for a minimal-length coset representative corresponding to the partition $\eta$, the diagram

$$\begin{array}{ccc} \mathscr{P}([|\eta|]) & \xrightarrow{\;p_c \circ p_d^{-1}\;} & \mathscr{P}([|\eta|]) \\ & {\scriptstyle g_d} \searrow \quad \swarrow {\scriptstyle g_c} & \\ & W & \end{array}$$

commutes, thus showing that the correspondence is independent of the choice of reduced word.

We show this by induction on $|\eta|$. The diagram commutes trivially if $|\eta| = 1$. Assume $\eta \subset \eta'$ with $|\eta'/\eta| = 1$, and the result holding for partitions with weight $|\eta|$. Let $g'$ be the corresponding minimal-length coset representative for $\eta'$. Choose reduced words $s, t$ for $g'$. Assume that for some $j$, $g' = \sigma_j g$. This is a safe assumption, since there exists an $\eta \subset \eta'$ satisfying this. Let $i_s, i_t$ be the first occurrences of $j$ in $s, t$ respectively. If $S \subset [|\eta'|]$ and $h = g_c \circ p_c \circ p_d^{-1}(S) \neq g_d(S) = h'$, then $S$ contains $i_s$. Furthermore, $\sigma_j h \neq \sigma_j h'$. But this is a contradiction, since $\sigma_j$ commutes with all of the generators occurring before $i_s$ in $s$ and $i_t$ in $t$, and the set $S - \{i_s\}$ makes the diagram commute by the following argument. We can identify $[|\eta'|] - \{i_s\}$ with $[|\eta|]$ by subtracting 1 from elements greater than $i_s$. Under this identification, the diagram commutes by the induction hypothesis, since the result holds for $\eta$.

We will hereafter assume that $b = d$ and define $p = p_d$. A compatible imbedding $f : \mathrm{Sd}(\lambda) \to \mathrm{Sd}(\mu)$ determines a function $\tilde{f} = p \circ f$.

Let $T \in SSYT(\lambda, \mu)$. If there exists an imbedding $\mathrm{Sd}(\lambda)$ in the upper left-hand corner of $\mathrm{Sd}(\mu)$, this corresponds under $p$ to a subsequence $c = b_{i_1} \ldots b_{i_{|\lambda|}}$ of $b$ that is a reduced word for $v$. Let $T_0$ be the tableau corresponding to this imbedding.

Note that a tableau can be transformed into $T_0$ by, starting at the top row, repeatedly subtracting 1 from the numbers in all of the boxes in the row (from left to right) until the number is minimal (i.e., equal to the row number). In the language of compatible imbeddings, this translates to finding the first box $B$ in the image of $f_T$ that is not in the image of $f_{T_0}$ that one finds by reading the rows from left to right and moving it diagonally one box up and to the left; that is, setting $r(f_{T'}(B)) = |B| - 1$, and repeating this until all boxes are in the image of $f_{T_0}$. We wish to show that $\tilde{f}_T(\mathrm{Sd}(\lambda))$ contains the indices of a subsequence of $b$ that is a reduced word of $v$ if and only if the same is true of $\tilde{f}_{T'}(\mathrm{Sd}(\lambda))$. This can be seen by noting that if $B \in T$ is the first box that is out of place, then there is no box immediately to the left of $B$, and no box immediately above $B$; this means that if one were to read off the word from left to right down the rows, any box between $(r(B)-1, co(B)-1)$ and $B$ corresponds to a number $k_i$ in the word $b$ with $|n - c(B) - k_i| > 1$, so $\sigma_{n-c(B)}$ commutes with all of the $\sigma_{k_i}$. Thus, transferring $B$ from $(r(B), co(B))$ to $(r(B)-1, co(B)-1)$ yields a reduced word for the same Weyl group element as $T$. Since every tableau in $SSYT(\lambda, \mu)$ can be transformed into $T_0$ in this fashion, each translates to a subsequence of $b$ that is a reduced word for $v$.

Having shown that every element of $SSYT(\lambda, \mu)$ corresponds to subsequence of $b$ that is a reduced word for $v$, we must construct a tableau from a subsequence $c = b_{i_1} \cdots b_{i_{|\lambda|}}$ that is a reduced word for $v$. Equivalently, we will construct a compatible imbedding $f_c : \mathrm{Sd}(\lambda) \to \mathrm{Sd}(\mu)$. Let $I = p^{-1}(c)$, and let $B \in \mathrm{Sd}(\lambda)$. Then, define $f_c(B)$ to be the box $C$ such that $c(C) = c(B)$, and there are $r(B) - 1$ boxes $D \in I$ with $c(D) = c(B)$ before $C$ when the boxes are read from left to right down the rows. $f_c$ is clearly injective, and $c(f_c(B)) = c(B)$ for all $B \in \mathrm{Sd}(\lambda)$. Thus conditions (1) and (2) for a compatible imbedding are satisfied. In this case, conditions (3) and (4) can be restated as

(3) If $B, C \in \mathrm{Sd}(\lambda)$, $co(B) \geq co(C)$, and there are more boxes $D \in I$ before $f_c(B)$ with $c(D) = c(B)$ than there are before $f_c(C)$ with $c(D) = c(C)$, then $r(f_c(B)) > r(f_c(C))$.

(4) If $B, C \in \mathrm{Sd}(\lambda)$, $c(B) < c(C)$, and there are as many boxes $D \in I$ before $f_c(C)$ with $c(D) = c(C)$ as there are before $f_c(B)$ with $c(D) = c(B)$, then $r(f_c(C)) \geq r(f_c(B))$.

This follows from Lemma 3.2: (3) follows from the fact that $f_c(B)$ occurs after $f_c(C)$ (so that $r(f_c(B)) \geq r(f_c(C))$) and, by the lemma, $0 < c(C) - c(B) + (co(f_c(B)) - co(f_c(C))) = (r(f_c(B)) - r(f_c(C)))$. (4) follows by the same fact. ∎

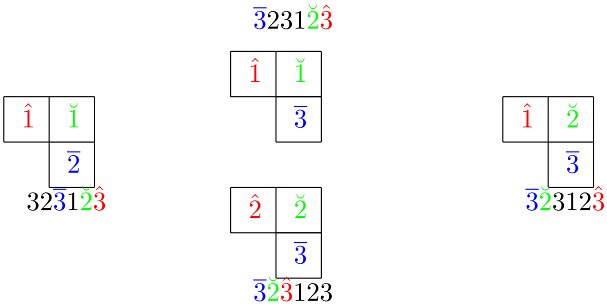



Figure 6: The correspondence between tableaux of shape (21) compatible with (321) and subsequences of a reduced word for (321) that are reduced words for (21).

*Proof (of Proposition 2.1, type C case):* In [B], a formula is given for certain functions of Kostant and Kumar in terms of positive roots. It is shown in [IN] that this formula also gives the restriction of a Schubert class to a fixed point in Grassmannians of types $B$, $C$, and $D$. If $s$ is an element of the Weyl group of type $C_n$, let $R(s)$ denote the set of reduced words of $s$. Choose Weyl group elements $v$ and $w$ that are minimal-length $W_P$-coset representatives, and let $b = b_1 \cdots b_n$ be a reduced word for $w$. Then

$$[X_v]|_w = \sum_{b_{i_1} \cdots b_{i_k} \in R(v)} r_{i_1}(b) \cdots r_{i_k}(b),$$

where $r_i(b) = \ldots \sigma_{b_{i-2}} \sigma_{b_{i-1}} \alpha_{b_i}$.

Let $\lambda$ be the strict partition corresponding to $v$, and $\mu$ the strict partition corresponding to $w$. We choose a particular reduced expression for $w$, namely

$$b = \prod_{i=1}^{l(\mu)} \prod_{j=-\mu_{l(\mu)+1-i}}^{-1} \sigma_{n+1+j}.$$

For $f$ a compatible imbedding, define $\tilde{f}$ as in the proof of Proposition 3.1. The result will follow easily from

**Claim:** If $T \in SSYT(\lambda, \mu)$ and $B \in T$, then $r_{\tilde{f}_T(B)}(b) = B|_\mu$.

*Proof (of Claim):* Let $T \in SSYT(\lambda, \mu)$, and choose $B \in T$. Then $B$ corresponds to an index $i = \tilde{f}_T(B)$. Suppose $b_i < n$. If $a_\mu(B) \neq 0$, then $b_i$ is preceded by the sequence $(b_{i-a_\mu(B)}, \ldots, b_{i-1})$, which is equal to $(b_i - a_\mu(B), \ldots, b_i - 1)$. $\sigma_{b_i - a_\mu(B)} \cdots \sigma_{b_i - 1} \alpha_{b_i} = \sum_{k=0}^{a_\mu(B)} \alpha_{b_i - k}$. Since there are no numbers smaller than $b_i - a_\mu(B)$ preceding the index $i - a_\mu(B)$, $\alpha_{i - a_\mu(B)}$ is the lowest root that occurs in $\cdots \sigma_{b_{i-1}} \alpha_{b_i}$, so $-t_{n-c(B)-a_\mu(B)} = -t_{n+1-\mu_{r(B)}}$ will occur in the resulting root.

At this point, our root is $\sum_{k=0}^{a_\mu(B)} \alpha_{b_i-k}$, and the numbers preceding the index $i-a_\mu(B)$ in $b$ are greater than or equal to $n+1-\mu_{r(B)}$. The top root index so far is $\alpha_{b_i}$. If there are no boxes beneath $B$ in the same column in its imbedding into $\mathrm{Sd}(\mu)$, then all numbers in $b$ preceding the index $i-a_\mu(B)$ are greater than $b_i+1$, and hence the Weyl group elements corresponding to these numbers have no effect on the result. Thus, in this case, $t_{b_i+1} = t_{n-c(B)}$ is the second term, and since $2n+1-(n-c(B)) = n+1+c(B)$, it is also equal to $-t_{n+1+c(B)}$, so the formula holds in this case.

Suppose $l_\mu(B) \neq 0$. Then there is a subsequence in $b$ preceding the index $i-a_\mu(B)$ consisting of $b_i+1, \ldots, b_i+l_\mu(B)$, each corresponding to a box beneath the image of $B$. The parts of the word corresponding to the boxes between this sequence and the index $i-a_\mu(B)$ leave the root unchanged, as they are either higher than the top root ($\alpha_{b_i+j}$, $1 \leq j \leq l_\mu(B)$) or correspond to the reflections of roots $\alpha_j$ such that $\alpha_{j-1}$ and $\alpha_{j+1}$ both occur in the sum, so that $\sigma_j(\alpha_{j-1}+\alpha_j+\alpha_{j+1}) = (\alpha_{j-1}+\alpha_j) - \alpha_j + (\alpha_{j+1}+\alpha_j) = \alpha_{j-1}+\alpha_j+\alpha_{j+1}$, and the rest of the roots are unaffected. Thus, roots up to $\alpha_{n-c(B)+l_\mu(B)}$ are added to the sum.

Now we have $\sum_{k=-l_\mu(B)}^{a_\mu(B)} \alpha_{b_i-k}$. If $l_\mu(B) \neq c(B)$, then we are done, and the top term is $t_{b_i+l_\mu(B)} = t_{n-c(B)+l_\mu(B)}$, and since $\mu_{|B|+c(B)} = 0$ and $2n+1-(n-c(B)+l_\mu(B)) = n+1+c(B)-l_\mu(B)$, the top term is $-t_{n+1+c(B)-l_\mu(B)}$, and the formula holds in that case.

If $l_\mu(B) = c(B)$, then $b_i + l_\mu(B) = n$, and so the remaining numbers in the word are $n-1, \ldots, n+1-\mu_{|B|+c(B)}$. When each of these $\sigma_j$'s are applied, it sends $\alpha_j \mapsto -\alpha_j$ and, if $j = n-1$, $\alpha_n \mapsto 2\alpha_{n-1}+\alpha_n$, and $2\alpha_{j+1} \mapsto 2(\alpha_j+\alpha_{j+1})$ otherwise, and $\alpha_{j-1} \mapsto \alpha_{j-1}+\alpha_j$. Thus each $\sigma_j$ puts a coefficient of 2 in front of $\alpha_k$ for $n+1-\mu_{|B|+c(B)} \leq k < n$. This has the effect of making the second term $-t_{n+1-\mu_{|B|+c(B)}+c(B)-l_\mu(B)}$, which finishes the proof for $b_i < n$.

If $b_i = n$, then only the terms in the "arm" portion of the row $B$ is mapped to in $\mu$ affect the root, and we are left with $\alpha_n + 2\sum_{k=1}^{a_\mu(B)} \alpha_{n-k}$, so the result is $-2t_{n-a_\mu(B)} = -2t_{n+1-\mu_{|B|}}$, as desired. □

To complete the proof, note that by Proposition 3.1, the tableaux in $SSYT(\lambda, \mu)$ correspond bijectively to subsequences of the chosen reduced word for $w$ that are reduced words for $v$, and by the Claim the product of the weights of the boxes in a given tableaux is equal to the term occuring in Billey's formula. ■

### 3.3 The odd orthogonal Grassmannian

This is the case $G = SO(2n+1, \mathbb{C})$. There is again essentially only one choice of parabolic subgroup, and $G/P = OGr(n, 2n+1)$.

In this case, $W$ and $W_P$ are the same as in the case of the Lagrangian Grassmannian. The root system, however, is of type $B_n$, generated by the same simple roots as type $C_n$, except that $\alpha_n = -t_n$. The action of $W$ is as follows:

$$\sigma_i \alpha_j = \begin{cases} -\alpha_j & \text{if } i = j \\ \alpha_j & \text{if } |i-j| > 1 \\ \alpha_i + \alpha_j & \text{if } i < n \text{ and } |i-j| = 1 \\ \alpha_{n-1} + 2\alpha_n & \text{if } i = n \text{ and } j = n-1. \end{cases}$$

Let $\varphi : \mathbb{R}^n - \{0\} \to \mathbb{R}^n - \{0\}$ be the function

$$\varphi(t) = \frac{2t}{(t,t)}.$$

Then

**Definition-Lemma 3.5.** Let $B^C|_\mu$ be the type $C$ weight of $B$ (Definition 3.3). Then the type $B$ weight of $B$ is defined to be

$$B|_\mu = \varphi(B^C|_\mu),$$

and this makes Proposition 2.1 true in the type $B$ case.

*Proof:* Let $c$ be a type $C$ root, $\alpha_i^C$ a type $C$ simple root, and $\sigma_i^C$ the corresponding type $C$ Weyl group element. $\varphi$ is known to map the root system of type $C_n$ onto the root system of type $B_n$, with the simple roots $\alpha_i^C$ mapped onto $\alpha_i$. $\varphi$ is also an involution. Furthermore, $\varphi$ is equivariant with respect to the Weyl group action: First, note that $|c-2\frac{(c,\alpha_i^C)}{(\alpha_i^C,\alpha_i^C)}\alpha_C^i|^2 = (c,c)-4\frac{(c,\alpha_i^C)^2}{(\alpha_i^C,\alpha_i^C)}+4\frac{(c,\alpha_i^C)^2}{(\alpha_i^C,\alpha_i^C)} = (c,c)$,

so

$$\begin{aligned}\varphi(c) &= \tfrac{2}{(c,c)}(c-2\tfrac{(c,\alpha_i^C)}{(\alpha_i^C,\alpha_i^C)}\alpha_i^C) \\ &= \tfrac{2c}{(c,c)} - \frac{2(c,\frac{2\alpha_i}{(\alpha_i,\alpha_i)})}{(c,c)}\alpha_i \\ &= \tfrac{2c}{(c,c)} - 2\frac{(\frac{2c}{(c,c)},\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i,\end{aligned}$$

so that $\varphi(\sigma_i^C c) = \sigma_i\varphi(c)$. ■

**Definition 3.6.** The function $r_i$ in type $B$ is given by

$$\begin{aligned}r_i(B,\lambda,\eta) &:= \delta_{n+1-i,n+1-\eta_{|B|}}(\eta_{|B|}-\lambda_{|B|}) \\ &+ \delta_{n+1-i,n-c(B)+l_\eta(B)}(1+\delta_{c(B),l_\eta(B)})(l_\eta(B)-l_\lambda(B)) \\ &+ (1-\delta_{c(B),0})\delta_{n+1-i,n+1-\eta_{|B|+c(B)}}\delta_{l_\lambda(B),c(B)}(\eta_{|B|+c(B)}-\lambda_{|B|+c(B)}).\end{aligned}$$

## 3.4 The even orthogonal Grassmannian

This is the case $G = SO(2n,\mathbb{C})$, where $G/P = OGr(n,2n)$. There are two choices of parabolic subgroup; each choice leads to a different torus action, and hence to a different weight formula. We have chosen one to use.

**Definition 3.7.** The weight of a box in type $D$ is given by

$$B|_\mu = t_{\mu_{|B|}} - (-1)^{\delta_{c(B),l_\mu(B)}\delta_{\mu_{|B|+c(B)+1},0}(|B|+c(B))} t_{1+c(B)-l_\mu(B)-\mu_{|B|+c(B)+1}},$$

where $t_k = -t_{1-k}$ if $k<1$.

**Definition 3.8.** The function $r_i$ in type $D$ is defined by

$$\begin{aligned}r_i(B,\lambda,\eta) &= \delta_{\eta_{|B|},i}(\eta_{|B|}-\lambda_{|B|}) + \delta_{i,c(B)+1-l_\eta(B)}(l_\eta(B)-l_\lambda(B)) \\ &+ \delta_{c(B),l_\lambda(B)}\delta_{\eta_{|B|+c(B)+1},i}(\eta_{|B|+c(B)+1}-\lambda_{|B|+c(B)+1}).\end{aligned}$$

The root system for $G$ is of type $D_n$, which has a different Weyl group from the other two cases. We pick a more convenient choice of simple roots:

$$\alpha_k = t_k - t_{k-1}$$

for $1<i$, and

$$\alpha_1 = t_1 + t_2.$$

The action of $W$ on the simple roots is as follows:

$$\sigma_i\alpha_j = \begin{cases} -\alpha_j & \text{if } i=j \\ \alpha_j & \text{if } 1\notin\{i,j\} \text{ and } |i-j|>1, \text{ or } \{i,j\}=\{1,2\} \\ \alpha_1+\alpha_3 & \text{if } \{i,j\}=\{1,3\} \\ \alpha_i+\alpha_j & \text{if } |i-j|=1 \text{ and } 1\notin\{i,j\}. \end{cases}$$

The group can again be described as a group of signed permutations of a set with $n$ elements, except that $\sigma_1$ both permutes and negates the first and second elements. The choice of parabolic subgroup $P$ that we will derive a weight formula for corresponds to the parabolic subgroup $W_P = \langle\sigma_1\rangle$. The minimal-length representatives of the $W_P$-cosets are again those permutations that produce decreasing sequences. There are only $2^{n-1}$ of these, and they correspond to strict partitions with all parts less than or equal to $n-1$.

**Lemma 3.3.** *Let $w \in W$, and let $u \in wW_P$ be a minimal-length coset representative. If $b$ is a reduced word for $u$, then $b$ is a reverse lattice word. Furthermore, $b$ ends with a 1, and 1's and 2's occur alternately, e.g.* 132431.

*Proof:* A modification of the proof of Lemma 3.2, together with the fact that $\sigma_1\sigma_3\sigma_1 = \sigma_3\sigma_1\sigma_3$, yields this lemma. ■

*Proof (of Proposition 2.1 in the type D case):* We again use the formula in [B] to prove this. We start with a particular reduced word $b$ for $\mu$: $b = b^{(l(\mu))}b^{(l(\mu)-1)}\cdots b^{(1)}$, where $b^{(2i-1)} = (\mu_{2i-1}+1, \mu_{2i-1}, \ldots, 3, 1)$ (or (1) if $\mu_{2i-1} = 1$) and $b^{(2i)} = (\mu_{2i}+1, \mu_{2i}, \ldots, 3, 2)$ (or (2) if $\mu_{2i} = 1$). This sequence is in bijective correspondence with the boxes in the shifted diagram of $\mu$, via the same map $p$ as in the proof of the Lagrangian Grassmannian formula. Shifted tableaux of type $D$ of shape $\lambda$ compatible with $\mu$ are in bijective correspondence with subsequences of $b$ that are reduced words for $\lambda$; this follows from the same argument as for the Lagrangian Grassmannian, coupled with Lemma 3.3. We must show that this correspondence gives the proper weight for the boxes.

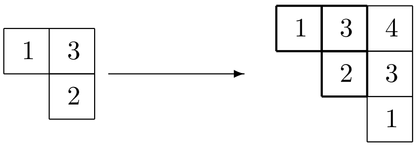



Figure 7: The unique compatible imbedding (21)→(321) of type D. The entries in the boxes correspond to Weyl group elements.

Let $T$ be a shifted tableau of type $D$ of shape $\lambda$ compatible with $\mu$, and let $B \in T$. The compatible imbedding corresponding to $T$, composed with $p$, maps $B$ to an index $i$ for an element in $b$. Assume $b_i \neq 1$. We start constructing a root $r$ at $r = \alpha_{b_i}$. $i$ falls in the $b^{(|B|)}$ part of $b$, so the remainder of $b^{(|B|)}$ is $(\mu_{|B|}+1, \mu_{|B|}, \ldots, b_i - 1)$. Applying all of these $\sigma$'s yields $\sum_{k=c(B)+2}^{\mu_{|B|}+1} \alpha_k$. If $b_i = 1$, then we have $\alpha_1 + \sum_{k=c(B)+3}^{\mu_{|B|}+1} \alpha_k$

Now, any number occuring before the index $i - a_\mu(B)$ is smaller than $\mu_{|B|}$, so only roots $\alpha_k$ can be added with $k < c(B)+1$. If $c(B) > l_\mu(B)$, then all of the roots between $c(B)+1-l_\mu(B)$ and $c(B)+1$ are added, and we have $r = \sum_{k=c(B)+1-l_\mu(B)}^{\mu_{|B|}} \alpha_i$. Then we get $r = t_{\mu_{|B|}} - t_{1+c(B)-l_\mu(B)}$, and we are finished.

If, on the other hand, $c(B) = l_\mu(B)$, then we have an $\alpha_1$ or $\alpha_2$ in the root, corresponding to when $|B| + c(B)$ is odd or even, respectively. If $\mu_{|B|+c(B)+1} = 0$, then the bottom term is $t_1$ if we have $\alpha_1$, and $-t_1$ otherwise, so the formula works in that case. Otherwise, we get the $\alpha_1$ or $\alpha_2$ that we don't have from row $|B| + c(B) + 1$. After this, we have $\alpha_1 + \alpha_2 + \cdots$. If $\mu_{|B|+c(B)+1} = 1$, then this gives a bottom term of $t_2$. If $\mu_{|B|+c(B)+1} > 1$, then the $\sigma_3$ produces $\alpha_1 + \alpha_2 + 2\alpha_3 + \cdots$, and every remaining box in the row puts a coefficient of 2 in front of $\alpha_{i+1}$, for box $i$. Thus we get a bottom term of $+t_{\mu_{|B|+c(B)+1}} = -t_{1-\mu_{|B|+c(B)+1}}$, which is the desired result. ■

# 4 Proof of the main theorem

**Proposition 4.1.** *With $\eta$ satisfying the conditions $\nu$ does in the statement of Theorem 1,*

$$B|_\eta - B|_\lambda = \sum_{i=1}^{n} r_i(B, \lambda, \eta)\alpha_i.$$

*Proof:* This will follow from examination of the proofs of the weight formulas. Consider the type

$C$ case; the root we end up with for the restriction to $\eta$ is

$$\sum_{k=-l_\eta(B)}^{a_\eta(B)} \alpha_{n-c(B)-k} + \delta_{l_\eta(B),c(B)} \sum_{k=1}^{\eta_{|B|+c(B)}} \alpha_{n-k},$$

and for $\lambda$ we have

$$\sum_{k=-l_\lambda(B)}^{a_\lambda(B)} \alpha_{n-c(B)-k} + \delta_{l_\lambda(B),c(B)} \sum_{k=1}^{\lambda_{|B|+c(B)}} \alpha_{n-k}.$$

Subtracting the second restriction from the first gives
$\sum_{k=-l_\eta(B)}^{-l_\lambda(B)-1} \alpha_{n-c(B)-k} + \sum_{k=a_\lambda(B)+1}^{a_\eta(B)} \alpha_{n-c(B)-k}$
$+ \delta_{l_\eta(B),c(B)} \sum_{k=1}^{\eta_{|B|+c(B)}} \alpha_{n-k} - \delta_{l_\lambda(B),c(B)} \sum_{k=1}^{\lambda_{|B|+c(B)-1}} \alpha_{n-k}.$
Recall the type $C$ definition:
$r_i(B,\lambda,\eta) := \delta_{n+1-i,n+1-\eta_{|B|}}(\eta_{|B|} - \lambda_{|B|})$
$+ \delta_{n+1-i,n-c(B)+l_\eta(B)}(l_\eta(B) - l_\lambda(B))$
$+ \delta_{n+1-i,n+1-\eta_{|B|+c(B)}}\delta_{l_\lambda(B),c(B)}(\eta_{|B|+c(B)} - \lambda_{|B|+c(B)}).$

Here we've subtracted $i$ from $n+1$, so that the $r_i(B,\lambda,\eta)$ measures the coefficient of the root $\alpha_{n+1-i}$ in $B|_\eta - B|_\lambda$. If the leg length of $B$ in $\eta$ is greater than the leg length of $B$ in $\lambda$, then we get a contribution of $\alpha_{n-c(B)+l_\eta(B)}$ to the difference, giving us the second term in $r_i$. If the arm length of $B$ is greater in $\eta$ than in $\lambda$, then we get a contribution of $\alpha_{n-a_\eta(B)-c(B)} = \alpha_{n+1-\eta_{|B|}}$.

If $l_\eta(B) \neq c(B)$, then we are finished. If $l_\lambda(B) = c(B)$, then $l_\eta(B) = c(B)$, and so we have a contribution of $\alpha_{n+1-\eta_{|B|+c(B)}}$ if $\eta_{|B|+c(B)} > \lambda_{|B|+c(B)}$. If $l_\eta(B) = c(B)$ but $l_\lambda(B) \neq c(B)$, then $\eta_{|B|+c(B)} = 1$, so we get a contribution of $\alpha_n$. Thus, $r_i(B,\lambda,\eta)$ is indeed the coefficient of $\alpha_{n+1-i}$ in $B|_\eta - B|_\lambda$, so the type $C$ case is finished.

For the type $B$ case, recall the definition
$r_i(B,\lambda,\eta) := \delta_{n+1-i,n+1-\eta_{|B|}}(\eta_{|B|} - \lambda_{|B|})$
$+ \delta_{n+1-i,n-c(B)+l_\eta(B)}(1 + \delta_{c(B),l_\eta(B)})(l_\eta(B) - l_\lambda(B))$
$+ (1 - \delta_{c(B),0})\delta_{n+1-i,n+1-\eta_{|B|+c(B)}}\delta_{l_\lambda(B),c(B)}(\eta_{|B|+c(B)} - \lambda_{|B|+c(B)}).$

The restrictions are the same in the type $B$ case as in the type $C$ case, except for the occasional division by two, and the roots are the same except that $\alpha_n^C = 2\alpha_n^B$. This is reflected in the second term by multiplication by two when $2\alpha_n^B = \alpha_n^C$ is obtained by a leg length increase, and the $\delta_{c(B),0}$ in the third term ensures that the coefficient is divided by 2 for an arm length change on a box with $c(B) = 0$.

For the type $D$ case, recall the definition
$r_i(B,\lambda,\eta) = \delta_{\eta_{|B|},i}(\eta_{|B|} - \lambda_{|B|}) + \delta_{i,c(B)+1-l_\eta(B)}(l_\eta(B) - l_\lambda(B))$
$+ \delta_{c(B),l_\lambda(B)}\delta_{\eta_{|B|+c(B)+1},i}(\eta_{|B|+c(B)+1} - \lambda_{|B|+c(B)+1}).$

From the previous arguments, the first term and second term are clear. For the third term, note that $\alpha_{\eta_{|B|+c(B)+1}}$ is added to the root if $c(B) = l_\lambda(B)$ and $\eta_{|B|+c(B)+1} > \lambda_{|B|+c(B)+1}$, because, as in the proof of the weight formula, $\alpha_{\eta_{|B|+c(B)+1}}$ has a coefficient of 2 instead of 1. ∎

**Proposition 4.2.** *Let $(x_i)$, $(y_i)$ be sequences with $m$ elements in commutative ring. Then*

$$\prod_{i=1}^m x_i - \prod_{i=1}^m y_i = \sum_{i=1}^m (x_i - y_i) \prod_{j=1}^{i-1} x_j \prod_{k=i+1}^m y_k.$$

*Proof:* The expression being summed is equal to

$$\prod_{j=1}^i x_j \prod_{k=i+1}^m y_k - \prod_{j=1}^{i-1} x_j \prod_{k=i}^m y_k.$$

The sum of this over $1 \leq j \leq m$ telescopes, leaving $\prod_{i=1}^m x_i - \prod_{i=1}^m y_i$. ∎

*Proof (of Theorem 1):* Since $c^{\lambda}_{\lambda\mu} = [X_{mu}]|_{\lambda}$, the theorem holds in the case of the triple $(\mu, \lambda, \lambda)$. For a triple $(\mu, \lambda, \nu)$ satisfying the outside corner condition and $|\nu| - |\lambda| = k$, assume the result holds for all $(\mu, \eta, \zeta)$ with $\lambda \subsetneq \eta \subset \zeta \subsetneq \nu$. Consider the desired result,

$$\sum_{\substack{T \in \overline{SSYT}(\mu,\nu,k) \\ P \in Paths(\lambda,\nu)}} r(T,P)w(T,P).$$

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