Equivariant cohomology of flag manifolds of types B, C, and D

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Preliminaries and definitions

We will make extensive use of the following definitions.

**Definition 1.** A strict partition $\lambda$ is a strictly decreasing sequence of positive integers. We will denote by $\mathcal{S}_n$ (for $n > 0$ an integer) the set of all strict partitions whose elements are less than or equal to $n$.

**Definition 2.** A flag is an increasing sequence of vector spaces $V_1 \subset V_2 \subset \cdots \subset V_m$ (over the same field) such that the dimension of $V_i$ is $i$.

**Grassmannians**

We start with some linear algebra.

**Definition 3.** Let $V$ be a vector space over a field $F$, $A \subset V$ a subspace, and $b : V \times V \to F$ a bilinear form. Then $A$ is said to be isotropic with respect to $b$ if for all $v, w \in A$, $b(v, w) = 0$.

**Definition 4.** Let $V$ be an even-dimensional vector space over a field $F$, and let $\omega : V \times V \to F$ be a skew-symmetric, nondegenerate bilinear form. Then a basis $\{v_1, \ldots, v_n, w_1, \ldots, w_n\}$ for $V$ is called $\omega$-isotropic if $\omega(v_i, v_j) = \omega(w_i, w_j) = 0$ and $\omega(v_i, w_j) = \delta_{ij}$ for all $i, j \leq n$.

**Proposition 1.** Let $V$ be a finite-dimensional vector space over a field $F$, and let $\omega : V \times V \to F$ be a skew-symmetric, nondegenerate bilinear form. Then there is an integer $n$ such that the dimension of $V$ is $2n$, and there exists an $\omega$-isotropic basis for $V$. If $A \subset V$ is a $2k$-dimensional subspace with an $\omega$-isotropic basis $\{v_1, \ldots, v_k, w_1, \ldots, w_k\}$, then this basis for $A$ can be extended to an $\omega$-isotropic basis for $V$.

Proof: Choose a nonzero $v_1 \in V$. Then, by nondegeneracy of the form, there is a vector $w_1$ with $\omega(v_1, w_1) = 1$. Since $\omega(v_1, w_1) \neq 0$, the set $\{v_1, w_1\}$ is linearly independent.

Assume inductively that for some $k > 0$ we have a linearly independent set $\{v_1, \ldots, v_k, w_1, \ldots, w_k\}$ that is an $\omega$-isotropic basis for a subspace of $V$ with
2k < \dim V$. Then there exists a vector $v \in V$ that is not in the subspace generated by $\{v_1, \ldots, v_k, w_1, \ldots, w_k\}$. Set
\[
v_{k+1} = v + \sum_{i=1}^{k} \omega(w_i, v)v_i - \sum_{i=1}^{k} \omega(v, v_i)w_i.
\]
Then by inspection $\omega(v_{k+1}, v_i) = \omega(v_{k+1}, w_i) = 0$ for all $i \leq k$. By nondegeneracy of the form, there exists a $w$ with $\omega(v_{k+1}, w) = 1$, and since $\omega(v_{k+1}, w) \neq 0$, $w$ is not in the subspace generated by $\{v_1, \ldots, v_{k+1}, w_1, \ldots, w_k\}$. Set
\[
w_{k+1} = w + \sum_{i=1}^{k} \omega(w_i, w)v_i - \sum_{i=1}^{k} \omega(v_i, w)w_i.
\]
Then the set $\{v_1, \ldots, v_{k+1}, w_1, \ldots, w_{k+1}\}$ is linearly independent, and for $i, j \leq k + 1$, $\omega(v_i, w_j) = \delta_{ij}$, and $\omega(v_i, v_j) = \omega(w_i, w_j) = 0$. Hence for each integer $m$ with $2m \leq \dim V$ we have constructed an $\omega$-isotropic basis for some subspace of $V$ of dimension $2m$. Furthermore, if $2m < \dim V$, we have shown that there exists a basis containing this one for a subspace of dimension $2m + 2$. Thus, there is some $n$ such that $2n = \dim V$, and therefore the stated basis exists. \[\blacksquare\]

Thus, there is essentially only one skew-symmetric, nondegenerate bilinear form on an even-dimensional complex vector space (and none on an odd-dimensional vector space). A similar statement is true for a symmetric bilinear form.

**Definition 5.** Let $V$ be a vector space over a field $F$, and let $\sigma : V \times V \to F$ be a symmetric, nondegenerate bilinear form. Then a basis $\{v_1, \ldots, v_n, w_1, \ldots, w_n\}$ for $V$ is called $\sigma$-isotropic if $\sigma(v_i, v_j) = \sigma(w_i, w_j) = 0$ and $\sigma(v_i, w_j) = \delta_{ij}$ for all $i, j \leq n$ (in case the dimension of $V$ is $2n$), and a basis $\{v_1, \ldots, v_n, w_1, \ldots, w_n, u\}$ for $V$ is called $\sigma$-isotropic if in addition $\sigma(u, v_i) = \sigma(u, w_i) = 0$ for all $i \leq n$, and $\sigma(u, u) = 1$ (in case the dimension of $V$ is $2n + 1$).

**Proposition 2.** Let $V$ be a finite-dimensional vector space over a field $F$ of characteristic not equal to 2 such that the map $F \to F$ given by $a \mapsto a^2$ is surjective, and let $\sigma : V \times V \to F$ be a symmetric, nondegenerate bilinear form. Then there exists a $\sigma$-isotropic basis for $V$. If $A \subset V$ is a subspace of dimension $2k$ with a $\sigma$-isotropic basis $\{v_1, \ldots, v_k, w_1, \ldots, w_k\}$, then this basis can be extended to a $\sigma$-isotropic basis for $V$.

Proof: Let $v \in V$ be a nonzero vector, and let $z \in V$ be such that $\sigma(v, z) = 1$. We wish to construct a vector $v_1$ such that $\sigma(v_1, v_1) = 0$. If this is true of either $v$ or $z$, then we are done with this step, so assume this is not the case. If $a \in F$, we can compute
\[
\sigma(v + az, v + az) = \sigma(v, v) + 2a\sigma(v, z) + a^2\sigma(z, z).
\]
Since $F$ is closed under taking square roots and $\sigma(z, z)$ is nonzero, we can solve $a^2\sigma(z, z) + 2a\sigma(v, z) + \sigma(v, v) = 0$ for $a$. Having done this, we set $v_1 = v + az$, so that $\sigma(v_1, v_1) = 0$.}

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By nondegeneracy of the form, there is a vector \( w \in V \) such that \( \sigma(v_1, w) = 1 \). We can compute
\[
\sigma(w + a'v_1, w + a'v_1) = \sigma(w, w) + 2a'\sigma(w, v_1),
\]
so if we set \( w_1 = w - \frac{\sigma(w, w)}{2\sigma(w, v_1)} v_1 \), we have \( \sigma(v_1, w_1) = 1 \) and \( \sigma(w_1, w_1) = 0 \).

Assume inductively that we have a linearly independent set \( \{v_1, \ldots, v_k, w_1, \ldots, w_k\} \) satisfying \( \sigma(v_i, v_j) = \sigma(w_i, w_j) = 0 \) and \( \sigma(v_i, w_j) = \delta_{ij} \) for all \( i, j \leq k \) that is not a basis. Let \( x \in V \) be a vector not in the subspace generated by \( \{v_1, \ldots, v_k, w_1, \ldots, w_k\} \). Then if we set
\[
u = x - \sum_{i=1}^{k} \sigma(x, w_i)v_i + \sum_{i=1}^{k} \sigma(x, v_i)w_i,
\]
we have that \( \sigma(v_i, u) = \sigma(w_i, u) = 0 \) for all \( i \leq k \). If \( \dim V = 2k + 1 \), we must have that \( \sigma(u, u) \neq 0 \), and hence we may assume \( \sigma(u, u) = 1 \), so the desired basis exists.

Otherwise, let \( y \in V \) be such that \( \sigma(u, y) = 1 \). As in the construction of \( u \) from \( x \) in the previous paragraph, we may assume that \( \sigma(v_i, y) = \sigma(w_i, y) = 0 \) for all \( i \leq k \), and this preserves \( \sigma(u, y) = 1 \). As in the first paragraph, construct \( v_{k+1} \) from \( u \) and \( y \) with \( \sigma(v_{k+1}, v_{k+1}) = 0 \), which preserves orthogonality with \( v_i, w_j \) for \( i \leq k \). Let \( w \in V \) be such that \( \sigma(v_{k+1}, w) = 1 \). We can construct a \( w' \) from \( v_{k+1} \) and \( w \) such that \( \sigma(w', w') = 0 \), as in the first paragraph. Finally, set
\[
w_{k+1} = w' - \sum_{i=1}^{k} \sigma(w', v_i)v_i + \sum_{i=1}^{k} \sigma(w', v_i)w_i,
\]
so that \( \{v_1, \ldots, v_{k+1}, w_1, \ldots, w_{k+1}\} \) satisfies \( \sigma(w_i, w_j) = \sigma(v_i, v_j) = 0 \) and \( \sigma(v_i, w_j) = \delta_{ij} \) for \( i, j \leq k + 1 \).

This inductive process will terminate by finite-dimensionality of \( V \), leaving a basis with the desired properties. ■

**Definition 6.** Let \( F \) be a field, and let \( \omega : F^{2n} \times F^{2n} \to F \) be a skew-symmetric, nondegenerate bilinear form. Then the set of all isotropic subspaces of \( F^{2n} \) (with respect to \( \omega \)) of dimension \( k \) is denoted by \( LGr(k, 2n) \) and is called a **Grassmannian** of type \( C \). \( LGr(n, 2n) \) is also referred to as the **Lagrangian Grassmannian**.

**Definition 7.** Let \( F \) be a field, and let \( \sigma : F^d \times F^d \to F \) be a symmetric, nondegenerate bilinear form. Then the set of all isotropic subspaces of \( F^d \) (with respect to \( \sigma \)) of dimension \( k \) is denoted by \( OGr(k, d) \) and is called an even or odd **orthogonal Grassmannian**, according to whether \( d \) is even or odd.

Note that the group \( Sp(2n, F) \) acts on \( LGr(k, 2n) \), taking isotropic subspaces to isotropic subspaces. The same is true of \( SO(d, F) \) on \( OGr(k, d) \). We will show that this action is always transitive for \( Sp(2n, F) \), and is transitive for the orthogonal Grassmannian if the map \( a \mapsto a^2 \) is surjective and char \( F \neq 2 \).
Lemma 1. The following statements hold:

(a) Let $V$ be a vector space of dimension $2n$ equipped with a skew-symmetric, nondegenerate bilinear form $\omega$, and let $A \subset V$ be a subspace of dimension $k$ that is isotropic with respect to $\omega$. Then there is an isotropic subspace $B$ of dimension $k$ such that there is an $\omega$-isotropic basis $\{v_1, \ldots, v_k, w_1, \ldots, w_k\}$ for $A + B$, where $\{v_1, \ldots, v_k\}$ is a basis for $A$ and $\{w_1, \ldots, w_k\}$ is a basis for $B$.

(b) Let $V$ be a vector space of dimension $2n$ or $2n + 1$ (over a field of characteristic not equal to 2 such that the map $F \to F, a \mapsto a^2$ is surjective) equipped with a symmetric, nondegenerate bilinear form $\sigma$, and let $A \subset V$ be an isotropic subspace of dimension $k$. Then there is an isotropic subspace $B$ of dimension $k$ such that there is a $\sigma$-isotropic basis $\{v_1, \ldots, v_k, w_1, \ldots, w_k\}$ for $A + B$, where $\{v_1, \ldots, v_k\}$ is a basis for $A$ and $\{w_1, \ldots, w_k\}$ is a basis for $B$.

In particular, $k \leq n$ in both cases.

Proof: (a) This is true if the dimension of $A$ is 1 by nondegeneracy of the form. Assume inductively that the statement holds for isotropic subspaces of $V$ of dimension $m - 1$, and let $A$ be an isotropic subspace of dimension $m$. Choose a basis $\{v_1, \ldots, v_m\}$ for $A$, and let $B'$ be the inductive complementary isotropic subspace to $\{v_1, \ldots, v_{m-1}\}$, with basis $\{w_1, \ldots, w_{m-1}\}$, $\omega(v_i, w_j) = \delta_{ij}$ for $i, j < m$. Then there is a basis $\{v_1, v_2, \ldots, v_{m-1}, v'_m\}$ for $A$ such that $\omega(v'_m, w_i) = 0$ for all $i \leq m$. The final vector in the basis is given by

$$v'_m = v_m + \sum_{i=1}^{m-1} \omega(w_i, v_m)v_i.$$ 

Now choose $w \in V$ with $\omega(v'_m, w) = 1$, and set

$$w_m = w - \sum_{i=1}^{m-1} \omega(v_i, w)w_i + \sum_{i=1}^{m-1} \omega(w_i, w)v_i.$$ 

Then $\{v_1, \ldots, v'_m, w_1, \ldots, w_m\}$ is the desired $\omega$-isotropic basis for $A + B$, where $B$ is the subspace generated by $\{w_1, \ldots, w_m\}$.

(b) This is very similar to the proof for (a). If the dimension of $A$ is 1, let $v_1$ be a generator for $A$. Then there is a $w \in V$ such that $\sigma(v_1, w) = 1$. Set

$$w_1 = w - \frac{1}{2}\omega(w, w)v_1.$$ 

Then $\{v_1, w_1\}$ is the desired $\sigma$-isotropic basis for $A + B$, where $B$ is the isotropic subspace generated by $w_1$.

Assume inductively that the statement holds for isotropic subspaces of $V$ of dimension $m - 1$, and let $A$ be an isotropic subspace of dimension $m$. Choose a
basis \( \{ v_1, \ldots, v_m \} \) for \( A \), and let \( B' \) be the inductive complementary isotropic subspace to \( \{ v_1, \ldots, v_{m-1} \} \), with basis \( \{ w_1, \ldots, w_{m-1} \} \). Let \( \sigma(v_i, w_j) = \delta_{ij} \), and \( \sigma(v_i, v_j) = \sigma(w_i, w_j) \) for \( i, j < m \). Then there is a basis \( \{ v_1', \ldots, v_{m-1}', v_m' \} \) for \( A \) such that \( \sigma(v_m', v_i) = 0 \) for all \( i \leq m \). The final vector in the basis is given by
\[
v_m' = v_m + \sum_{i=1}^{m-1} \sigma(w_i, v_m)v_i.
\]

Now choose \( w \in V \) with \( \sigma(v_m', w) = 1 \), and set
\[
w_m = w - \sum_{i=1}^{m-1} \sigma(v_i, w)w_i + \sum_{i=1}^{m-1} \sigma(w_i, w)v_i.
\]

Then \( \sigma(w_i, w_m) = \sigma(v_i, w_m) = 0 \) for \( i < m \). \( w_m \) is not yet the vector we want, for it is possible that \( \sigma(w_m, w_m) \neq 0 \). However, if we set
\[
w_m' = w_m - \frac{1}{2} \sigma(w_m, w_m)v_m',
\]
then \( \{ v_1, \ldots, v_m', w_1, \ldots, w'_m \} \) is the desired \( \sigma \)-isotropic basis for \( A + B \), where \( B \) is the subspace generated by \( \{ w_1, \ldots, w_m \} \).

**Corollary 1.** The action of \( \text{Sp}(2n, F) \) on \( LGr(k, 2n) \) is transitive. The action of \( \text{SO}(d, F) \) on \( OGr(k, d) \) is transitive (if every element of \( F \) has a square root and \( \text{char } F \neq 2 \)).

**Proof:** Let \( \mu : F^d \times F^d \to F \) be a symmetric, or skew-symmetric, nondegenerate bilinear form. Let \( A, A' \subset V \) be isotropic subspaces of dimension \( k \). Let \( B, B' \) be respective complementary isotropic subspaces as in Lemma 1 where \( A + B \) and \( A' + B' \) have respective \( \mu \)-isotropic bases \( \{ v_1, \ldots, v_k, w_1, \ldots, w_k \} \) and \( \{ v_1', \ldots, v_k', w_1', \ldots, w_k' \} \). Let \( M \) be a linear transformation such that \( Mv_i = v_i' \) and \( Mw_i = w_i' \). This is a well-defined map from \( A + B \) to \( A' + B' \). By Propositions 1 and 2, the chosen \( \mu \)-isotropic bases for \( A + B \) and \( A' + B' \) extend to \( \mu \)-isotropic bases for \( F^d \). Therefore, there is a linear transformation \( M' \) that is equal to \( M \) on \( A + B \), taking \( A \) to \( A' \), that preserves \( \mu \). Since \( A \) and \( A' \) were arbitrary, we have the result.

**Cell decompositions of isotropic Grassmannians**

**Type C**

From now on we work over the field of complex numbers.

The Grassmannian \( LGr(k, 2n) \) is the set of \( k \)-dimensional isotropic subspaces of \( \mathbb{C}^{2n} \) with respect to a skew-symmetric, nondegenerate bilinear form \( \omega \). Choose an \( \omega \)-isotropic basis \( \{ v_1, \ldots, v_n, w_1, \ldots, w_n \} \) and reorder it as \( \{ v_1, \ldots, v_n, w_n, \ldots, w_1 \} \). We will describe a decomposition of \( LGr(k, 2n) \) into disjoint subspaces homeomorphic to \( \mathbb{C}^m \) for some \( m \) called **Schubert cells**, which will turn out to be the
orbits of the action of the Borel subgroup $B \subset Sp(2n, \mathbb{C})$ consisting of upper triangular matrices with respect to this basis.

Let $F = F_1 \subset \cdots \subset F_{2n}$ be the flag $(v_1) \subset (v_1, v_2) \subset \cdots \subset (v_1, \ldots, v_n, w_n, \ldots, w_1)$. Let $(\lambda, \lambda')$ be a pair of strict partitions such that $\lambda_i \neq \lambda'_j$ for all $i, j$, and $l(\lambda) + l(\lambda') = k$. Then the Schubert cell $X^0_{\lambda, \lambda'} = X^0_{\lambda, \lambda'}(F)$ is defined by

$$\dim(E \cap F_j) = i \text{ if } n + 1 - \lambda_i \leq j < n + 1 - \lambda_{i+1},$$

and

$$\dim(E \cap F_j) = k - i \text{ if } n + \lambda'_{i+1} < j \leq n + \lambda'_i.$$  

From their definition, it is clear that upper triangular elements of the symplectic group leave these Schubert cells invariant.

**Shifted tableaux and Weyl groups**