Equivariant cohomology of flag manifolds of types B, C, and D

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Preliminaries and definitions

We will make extensive use of the following definitions.

Definition 1. A **strict partition** λ is a strictly decreasing sequence of positive integers. We will denote by $\mathcal{S}_n$ (for $n > 0$ an integer) the set of all strict partitions whose elements are less than or equal to n.

Definition 2. A **flag** is an increasing sequence of vector spaces $V_1 \subset V_2 \subset \cdots \subset V_m$ (over the same field) such that the dimension of V_i is i.

Grassmannians

We start with some linear algebra.

Definition 3. Let V be a vector space over a field F, $A \subset V$ a subspace, and $b : V \times V \to F$ a bilinear form. Then A is said to be **isotropic** with respect to b if for all $v, w \in A$, $b(v, w) = 0$.

Definition 4. Let V be an even-dimensional vector space over a field F, and let $\omega : V \times V \to F$ be a skew-symmetric, nondegenerate bilinear form. Then a basis $\{v_1, \ldots, v_n, w_1, \ldots, w_n\}$ for V is called **ω-isotropic** if $\omega(v_i, v_j) = \omega(w_i, w_j) = 0$ and $\omega(v_i, w_j) = \delta_{ij}$ for all $i, j \leq n$.

Proposition 1. *Let V be a finite-dimensional vector space over a field F, and let $\omega : V \times V \to F$ be a skew-symmetric, nondegenerate bilinear form. Then there is an integer n such that the dimension of V is $2n$, and there exists an ω-istropic basis for V. If $A \subset V$ is a $2k$-dimesional subspace with an ω-isotropic basis $\{v_1, \ldots, v_k, w_1, \ldots, w_k\}$, then this basis for A can be extended to an ω-isotropic basis for V.*

Proof: Choose a nonzero $v_1 \in V$. Then, by nondegeneracy of the form, there is a vector w_1 with $\omega(v_1, w_1) = 1$. Since $\omega(v_1, w_1) \neq 0$, the set $\{v_1, w_1\}$ is linearly independent.

Assume inductively that for some $k > 0$ we have a linearly independent set $\{v_1, \ldots, v_k, w_1, \ldots, w_k\}$ that is an ω-isotropic basis for a subspace of V with

$2k < \dim V$. Then there exists a vector $v \in V$ that is not in the subspace generated by $\{v_1, \ldots, v_k, w_1, \ldots, w_k\}$. Set

$$v_{k+1} = v + \sum_{i=1}^{k} \omega(w_i, v)v_i - \sum_{i=1}^{k} \omega(v_i, v)w_i.$$

Then by inspection $\omega(v_{k+1}, v_i) = \omega(v_{k+1}, w_i) = 0$ for all $i \leq k$. By nondegeneracy of the form, there exists a w with $\omega(v_{k+1}, w) = 1$, and since $\omega(v_{k+1}, w) \neq 0$, w is not in the subspace generated by $\{v_1, \ldots, v_{k+1}, w_1, \ldots, w_k\}$. Set

$$w_{k+1} = w + \sum_{i=1}^{k} \omega(w_i, w)v_i - \sum_{i=1}^{k} \omega(v_i, w)w_i.$$

Then the set $\{v_1, \ldots, v_{k+1}, w_1, \ldots, w_{k+1}\}$ is linearly independent, and for $i, j \leq k+1$, $\omega(v_i, w_j) = \delta_{ij}$, and $\omega(v_i, v_j) = \omega(w_i, w_j) = 0$. Hence for each integer m with $2m \leq \dim V$ we have constructed an ω-isotropic basis for some subspace of V of dimension $2m$. Furthermore, if $2m < \dim V$, we have shown that there exists a basis containing this one for a subspace of dimension $2m + 2$. Thus, there is some n such that $2n = \dim V$, and therefore the stated basis exists. ■

Thus, there is essentially only one skew-symmetric, nondegenerate bilinear form on an even-dimensional complex vector space (and none on an odd-dimensional vector space). A similar statement is true for a symmetric bilinear form.

Definition 5. Let V be a vector space over a field F, and let $\sigma : V \times V \to F$ be a symmetric, nondegenerate bilinear form. Then a basis $\{v_1, \ldots, v_n, w_1, \ldots, w_n\}$ for V is called **σ-isotropic** if $\sigma(v_i, v_j) = \sigma(w_i, w_j) = 0$ and $\sigma(v_i, w_j) = \delta_{ij}$ for all $i, j \leq n$ (in case the dimension of V is $2n$), and a basis $\{v_1, \ldots, v_n, w_1, \ldots, w_n, u\}$ for V is called σ-isotropic if in addition $\sigma(u, v_i) = \sigma(u, w_i) = 0$ for all $i \leq n$, and $\sigma(u, u) = 1$ (in case the dimension of V is $2n + 1$).

Proposition 2. *Let V be a finite-dimensional vector space over a field F of characteristic not equal to 2 such that the map $F \to F$ given by $a \mapsto a^2$ is surjective, and let $\sigma : V \times V \to F$ be a symmetric, nondegenerate bilinear form. Then there exists a σ-isotropic basis for V. If $A \subset V$ is a subspace of dimension $2k$ with a σ-isotropic basis $\{v_1, \ldots, v_k, w_1, \ldots, w_k\}$, then this basis can be extended to a σ-isotropic basis for V.*

Proof: Let $v \in V$ be a nonzero vector, and let $z \in V$ be such that $\sigma(v, z) = 1$. We wish to construct a vector v_1 such that $\sigma(v_1, v_1) = 0$. If this is true of either v or z, then we are done with this step, so assume this is not the case. If $a \in F$, we can compute

$$\sigma(v + az, v + az) = \sigma(v, v) + 2a\sigma(v, z) + a^2\sigma(z, z).$$

Since F is closed under taking square roots and $\sigma(z, z)$ is nonzero, we can solve $a^2\sigma(z, z) + 2a\sigma(v, z) + \sigma(v, v) = 0$ for a. Having done this, we set $v_1 = v + az$, so that $\sigma(v_1, v_1) = 0$.

By nondegeneracy of the form, there is a vector $w \in V$ such that $\sigma(v_1, w) = 1$. We can compute

$$\sigma(w + a'v_1, w + a'v_1) = \sigma(w, w) + 2a'\sigma(w, v_1),$$

so if we set $w_1 = w - \frac{\sigma(w,w)}{2\sigma(w,v_1)} v_1$, we have $\sigma(v_1, w_1) = 1$ and $\sigma(w_1, w_1) = 0$.

Assume inductively that we have a linearly independent set $\{v_1, \ldots, v_k, w_1, \ldots, w_k\}$ satisfying $\sigma(v_i, v_j) = \sigma(w_i, w_j) = 0$ and $\sigma(v_i, w_j) = \delta_{ij}$ for all $i, j \leq k$ that is not a basis. Let $x \in V$ be a vector not in the subspace generated by $\{v_1, \ldots, v_k, w_1, \ldots, w_k\}$. Then if we set

$$u = x - \sum_{i=1}^{k} \sigma(x, w_i) v_i + \sum \sigma(x, v_i) w_i,$$

we have that $\sigma(v_i, u) = \sigma(w_i, u) = 0$ for all $i \leq k$. If dim $V = 2k + 1$, we must have that $\sigma(u, u) \neq 0$, and hence we may assume $\sigma(u, u) = 1$, so the desired basis exists.

Otherwise, let $y \in V$ be such that $\sigma(u, y) = 1$. As in the construction of u from x in the previous paragraph, we may assume that $\sigma(v_i, y) = \sigma(w_i, y) = 0$ for all $i \leq k$, and this preserves $\sigma(u, y) = 1$. As in the first paragraph, construct v_{k+1} from u and y with $\sigma(v_{k+1}, v_{k+1}) = 0$, which preserves orthogonality with v_i, w_i for $i \leq k$. Let $w \in V$ be such that $\sigma(v_{k+1}, w) = 1$. We can construct a w' from v_{k+1} and w such that $\sigma(w', w') = 0$, as in the first paragraph. Finally, set

$$w_{k+1} = w' - \sum_{i=1}^{k} \sigma(w', w_i) v_i + \sum_{i=1}^{k} \sigma(w', v_i) w_i$$

so that $\{v_1, \ldots, v_{k+1}, w_1, \ldots, w_{k+1}\}$ satisfies $\sigma(w_i, w_j) = \sigma(v_i, v_j) = 0$ and $\sigma(v_i, w_j) = \delta_{ij}$ for $i, j \leq k + 1$.

This inductive process will terminate by finite-dimensionality of V, leaving a basis with the desired properties. ■

Definition 6. Let F be a field, and let $\omega : F^{2n} \times F^{2n} \to F$ be a skew-symmetric, nondegenerate bilinear form. Then the set of all isotropic subspaces of F^{2n} (with respect to ω) of dimension k is denoted by $LGr(k, 2n)$ and is called a **Grassmannian** of type C. $LGr(n, 2n)$ is also referred to as the **Lagrangian Grassmannian**.

Definition 7. Let F be a field, and let $\sigma : F^d \times F^d \to F$ be a symmetric, nondegenerate bilinear form. Then the set of all isotropic subspaces of F^d (with respect to σ) of dimension k is denoted by $OGr(k, d)$ and is called an even or odd **orthogonal Grassmannian**, according to whether d is even or odd.

Note that the group $Sp(2n, F)$ acts on $LGr(k, 2n)$, taking isotropic subspaces to isotropic subspaces. The same is true of $SO(d, F)$ on $OGr(k, d)$. We will show that this action is always transitive for $Sp(2n, F)$, and is transitive for the orthogonal Grassmannian if the map $a \mapsto a^2$ is surjective and char $F \neq 2$.

Lemma 1. *The following statements hold:*

(a) *Let V be a vector space of dimension $2n$ equipped with a skew-symmetric, nondegenerate blinear form ω, and let $A \subset V$ be a subspace of dimension k that is isotropic with respect to ω. Then there is an isotropic subspace B of dimension k such that there is an ω-isotropic basis $\{v_1, \ldots, v_k, w_1, \ldots, w_k\}$ for $A+B$, where $\{v_1, \ldots, v_k\}$ is a basis for A and $\{w_1, \ldots, w_k\}$ is a basis for B.*

(b) *Let V be a vector space of dimension $2n$ or $2n+1$ (over a field of characteristic not equal to 2 such that the map $F \to F$, $a \mapsto a^2$ is surjective) equipped with a symmetric, nondegenerate bilinear form σ, and let $A \subset V$ be an isotropic subspace of dimension k. Then there is an isotropic subspace B of dimension k such that there is a σ-isotropic basis $\{v_1, \ldots, v_k, w_1, \ldots, w_k\}$ for $A+B$, where $\{v_1, \ldots, v_k\}$ is a basis for A and $\{w_1, \ldots, w_k\}$ is a basis for B.*

In particular, $k \leq n$ in both cases.

Proof: (a) This is true if the dimension of A is 1 by nondegeneracy of the form. Assume inductively that the statement holds for isotropic subspaces of V of dimension $m-1$, and let A be an isotropic subspace of dimension m. Choose a basis $\{v_1, \ldots, v_m\}$ for A, and let B' be the inductive complementary isotropic subspace to $\{v_1, \ldots, v_{m-1}\}$, with basis $\{w_1, \ldots, w_{m-1}\}$, $\omega(v_i, w_j) = \delta_{ij}$ for $i, j < m$. Then there is a basis $\{v_1, v_2, \ldots, v_{m-1}, v'_m\}$ for A such that $\omega(v'_m, w_i) = 0$ for all $i \leq m$. The final vector in the basis is given by

$$v'_m = v_m + \sum_{i=1}^{m-1} \omega(w_i, v_m) v_i.$$

Now choose $w \in V$ with $\omega(v'_m, w) = 1$, and set

$$w_m = w - \sum_{i=1}^{m-1} \omega(v_i, w) w_i + \sum_{i=1}^{m-1} \omega(w_i, w) v_i.$$

Then $\{v_1, \ldots, v'_m, w_1, \ldots, w_m\}$ is the desired ω-isotropic basis for $A+B$, where B is the subspace generated by $\{w_1, \ldots, w_m\}$.

(b) This is very similar to the proof for (a). If the dimension of A is 1, let v_1 be a generator for A. Then there is a $w \in V$ such that $\sigma(v_1, w) = 1$. Set

$$w_1 = w - \frac{1}{2}\omega(w, w) v_1.$$

Then $\{v_1, w_1\}$ is the desired σ-isotropic basis for $A+B$, where B is the isotropic subspace generated by w_1.

Assume inductively that the statement holds for isotropic subspaces of V of dimension $m-1$, and let A be an isotropic subspace of dimension m. Choose a basis $\{v_1, \ldots, v_m\}$ for A, and let B' be the inductive complementary isotropic subspace to $\{v_1, \ldots, v_{m-1}\}$, with basis $\{w_1, \ldots, w_{m-1}\}$, $\sigma(v_i, w_j) = \delta_{ij}$, and $\sigma(v_i, v_j) = \sigma(w_i, w_j)$ for $i, j < m$. Then there is a basis $\{v_1, v_2, \ldots, v_{m-1}, v'_m\}$ for A such that $\sigma(v'_m, w_i) = 0$ for all $i \leq m$. The final vector in the basis is given by

$$v'_m = v_m + \sum_{i=1}^{m-1} \sigma(w_i, v_m) v_i.$$

Now choose $w \in V$ with $\sigma(v'_m, w) = 1$, and set

$$w_m = w - \sum_{i=1}^{m-1} \sigma(v_i, w) w_i + \sum_{i=1}^{m-1} \sigma(w_i, w) v_i.$$

Then $\sigma(w_i, w_m) = \sigma(v_i, w_m) = 0$ for $i < m$. w_m is not yet the vector we want, for it is possible that $\sigma(w_m, w_m) \neq 0$. However, if we set

$$w'_m = w_m - \frac{1}{2}\sigma(w_m, w_m) v'_m,$$

then $\{v_1, \ldots, v'_m, w_1, \ldots, w'_m\}$ is the desired σ-isotropic basis for $A + B$, where B is the subspace generated by $\{w_1, \ldots, w_m\}$. ■

Corollary 1. *The action of $Sp(2n, F)$ on $LGr(k, 2n)$ is transitive. The action of $SO(d, F)$ on $OGr(k, d)$ is transitive (if every element of F has a square root and char $F \neq 2$).*

Proof: Let $\mu : F^d \times F^d \to F$ be a symmetric, or skew-symmetric, nondegenerate bilinear form. Let $A, A' \subset V$ be isotropic subspaces of dimension k. Let B, B' be respective complementary isotropic subspaces as in Lemma 1 where $A+B$ and $A'+B'$ have respective μ-isotropic bases $\{v_1, \ldots, v_k, w_1, \ldots, w_k\}$ and $\{v'_1, \ldots, v'_k, w'_1, \ldots, w'_k\}$. Let M be a linear transformation such that $Mv_i = v'_i$ and $Mw_i = w'_i$. This is a well-defined map from $A + B$ to $A' + B'$. By Propositions 1 and 2, the chosen μ-isotropic bases for $A + B$ and $A' + B'$ extend to μ-isotropic bases for F^d. Therefore, there is a linear transformation M' that is equal to M on $A + B$, taking A to A', that preserves μ. Since A and A' were arbitrary, we have the result. ■

Cell decompositions of isotropic Grassmannians

Type C

From now on we work over the field of complex numbers.

The Grassmannian $LGr(k, 2n)$ is the set of k-dimensional isotropic subspaces of $\mathbb{C}^{2n}$ with respect to a skew-symmetric, nondegenerate bilinear form ω. Choose an ω-isotropic basis $\{v_1, \ldots, v_n, w_1, \ldots, w_n\}$ and reorder it as $\{v_1, \ldots, v_n, w_n, \ldots, w_1\}$. We will describe a decomposition of $LGr(k, 2n)$ into disjoint subspaces homeomorphic to $\mathbb{C}^m$ for some m called **Schubert cells**, which will turn out to be the orbits of the action of the Borel subgroup $B \subset Sp(2n, \mathbb{C})$ consisting of upper triangular matrices with respect to this basis.

Let $F = F_1 \subset \cdots \subset F_{2n}$ be the flag $\langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \cdots \subset \langle v_1, \ldots, v_n, w_n, \ldots, w_1 \rangle$. Let (λ, λ') be a pair of strict partitions such that $\lambda_i \neq \lambda'_j$ for all i, j, and $l(\lambda) + l(\lambda') = k$. Then the Schubert cell $X^0_{\lambda,\lambda'} = X^0_{\lambda,\lambda'}(F)$ is defined by

$$\dim(E \cap F_j) = i \text{ if } n + 1 - \lambda_i \leq j < n + 1 - \lambda_{i+1},$$

and

$$\dim(E \cap F_j) = k - i \text{ if } n + \lambda'_{i+1} < j \leq n + \lambda'_i.$$

From their definition, it is clear that upper triangular elements of the symplectic group leave these Schubert cells invariant.

Shifted tableaux and Weyl groups