29 June 2007

Quiz 12: Math 135, Section C7

Recall that the Mean Value Theorem for derivatives: If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists some c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Let $f(x) = x^2$ on the interval $[3, 6]$.

1. Find some value c in $[3, 6]$ that satisfies the conclusion of the Mean Value Theorem.

Let $g(x) = \frac{x}{x^2+1}$.

2. What is $g'(x)$?

3. On what intervals is g increasing? Decreasing?

4. On what intervals is g concave up? Concave down?

5. On the back of this sheet, sketch g.

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1. We need

$$f'(c) = \frac{f(6) - f(3)}{6 - 3}$$

or

$$2c = \frac{36 - 9}{3} = 9$$

So $c = 4.5$.

2. By the quotient rule,

$$g'(x) = \frac{(1)(x^2 + 1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$$

3. Our first-order critical points are ± 1, and a moment's thought shows that g is decreasing when $x < -1$ or $x > 1$ and increasing when $-1 < x < 1$.

4. We need the second derivative:

$$\begin{aligned} g''(x) &= \frac{(-2x)(x^2+1)^2 - (1-x^2)2(x^2+1)(2x)}{(x^2+1)^4} \\ &= \frac{-2x(x^2+1)(x^2+1+2(1-x^2))}{(x^2+1)^4} \\ &= \frac{-2x(x^2+1)(3-x^2)}{(x^2+1)^4} \end{aligned}$$

So our second order critical points are 0 and $\pm\sqrt{3}$. A moment's thought shows that g is concave up when $-\sqrt{3} < x < 0$ and $x > \sqrt{3}$ and concave down when $x < -\sqrt{3}$ and $0 < x < \sqrt{3}$.

5.