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Quiz 12: Section 10.2, Problem 10

Calculate S_3, S_4, and S_5 and then find the sum $S = \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$ using the identity

$$\frac{1}{4n^2 - 1} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

$$S_3 = \frac{1}{2}\left(1 - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right) = \frac{3}{7}$$
$$S_4 = \frac{1}{2}\left(1 - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right) + \frac{1}{2}\left(\frac{1}{7} - \frac{1}{9}\right) = \frac{4}{9}$$
$$S_5 = \frac{1}{2}\left(1 - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right) + \frac{1}{2}\left(\frac{1}{7} - \frac{1}{9}\right) + \frac{1}{2}\left(\frac{1}{9} - \frac{1}{11}\right) = \frac{5}{11}$$

Therefore, we see that $S_n = \frac{n}{2n+1}$ and so

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$$