9 June 2008

Quiz 5: Section 7.4, Problem 23

Integrate:

$$\int \frac{x^2}{\sqrt{x^2+1}}\,dx$$

We use the trigonometric substitution

$$x = \tan\theta$$
$$dx = \sec^2\theta\; d\theta$$

When we make the substitution, we get

$$\begin{aligned}
\int \frac{x^2}{\sqrt{x^2+1}}\,dx &= \int \frac{\tan^2\theta\sec^2\theta\; d\theta}{\sqrt{\tan^2\theta+1}} \\
&= \int \tan^2\theta\sec\theta\; d\theta \\
&= \int (\sec^2\theta-1)\sec\theta\; d\theta \\
&= \int \sec^3\theta\; d\theta - \int \sec\theta\; d\theta \\
&= \frac{\sec\theta\tan\theta}{2} + \frac{1}{2}\int \sec\theta\; d\theta - \int \sec\theta\; d\theta \\
&= \frac{\sec\theta\tan\theta}{2} - \frac{1}{2}\int \sec\theta\; d\theta \\
&= \frac{\sec\theta\tan\theta}{2} - \frac{1}{2}\ln|\sec\theta+\tan\theta| + C \\
&= \frac{\sec\tan^{-1}x\tan\tan^{-1}x}{2} - \frac{1}{2}\ln|\sec\tan^{-1}x+\tan\tan^{-1}x| + C
\end{aligned}$$

By looking at the appropriate right triangle, it can be found that $\sec\tan^{-1}x = \sqrt{x^2+1}$, so our final answer is

$$\int \frac{x^2}{\sqrt{x^2+1}}\,dx = \frac{\sqrt{x^2+1}\cdot x}{2} - \frac{1}{2}\ln|\sqrt{x^2+1}\cdot x + x| + C$$