Workshop Four: How to Count

1 Rules for the Workshop

This workshop will be done in randomly-assigned groups of three or four. Each student will get a copy of this workshop, but each group is to turn in only one workshop. It should be neat, clear, and concise. Show all mathematical work. This workshop is intended to be done in-class, but if you need time over the weekend to finish, come talk to me and get my permission.

By the way, read everything. You're going to be lost quickly if you don't take the time to read the introduction and everything else associated with these problems.

2 Introduction

We will focus on one aspect of power series, which is its use to derive closed-form formulas for sequences of numbers. The sequence we will focus on is the famous *Fibonacci Numbers*, named after Leonardo of Pisa, also known as Fibonacci. The original story is about rabbit populations, but the Fibonacci numbers are defined as follows:

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_n &= F_{n-1} + F_{n-2} \text{ for } n > 1 \end{aligned}$$

Write the next five numbers in the Fibonacci sequence:

The problem with this definition is that it's not *closed-form*, meaning this isn't a formula that depends on n and n alone. If you wanted to find $F_{1,000,000}$ using the recursive definition, you would need to find $F_0, F_1, F_2, \ldots, F_{999,999}$ first. That is quite inefficient! However, there *is* a closed-form formula for the Fibonacci sequence, called *Binet's Formula:*

$$F_n = \frac{1}{\sqrt{5}}\varphi^n - \frac{1}{\sqrt{5}}(\varphi')^n$$

φ is called the *Golden Ratio*, and is defined as $\varphi = \frac{1+\sqrt{5}}{2}$. φ' is a number similar to the golden ratio, and is defined by $\frac{1-\sqrt{5}}{2}$. In long form, Binet's Formula is

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

You may be worried that there are a lot of radicals in this equation for a sequence of *integers*, but try out a few values of n to convince yourself that things are working out! Power series are the key to finding Binet's Formula, so we will start with that.

3 The Power of Power Series

We spent some time with Taylor Polynomials, and power series are very similar objects. Simply put, a power series (centered at 0) is a function of the form

$$G(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

You can think of a power series as an "infinite polynomial". Notice that there is an underlying *sequence of coefficients* $\{a_0, a_1, \ldots\}$. One of the nicest things about power series is that you can manipulate them like you would any algebraic equation. For example, we can write

$$G(2x) = a_0 + a_1(2x) + a_2(2x)^2 + \cdots = a_0 + 2a_1 x + 4a_2 x^2 + \cdots = \sum_{k=0}^{\infty} 2^k a_k x^k$$

So this power series has its sequence of coefficients as $\{a_0, 2a_1, 4a_2, \ldots\}$. Similarly, we can write

$$2G(x) = 2a_0 + 2a_1 x + 2a_2 x^2 + \cdots = \sum_{k=0}^{\infty} 2a_k x^k$$

and this power series has a sequence of coefficients $\{2a_0, 2a_1, 2a_2, \ldots\}$. For this workshop, the only power series you will need to know is the following:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k$$

What is the sequence of coefficients for this power series?

What is the power series and sequence of coefficients for the function $\frac{1}{1-bx}$ for any number b?

4 Initial Steps

To start off, we want to express the Fibonacci Sequence as a power series - we shall call this new function $f(x)$:

$$f(x) = 0 + x + x^2 + 2x^3 + 3x^4 + 5x^5 + \cdots = \sum_{k=0}^{\infty} F_k x^k$$

We want to use our recursive formula, but that only works starting at $k = 2$. That means we need to separate the power series slightly:

$$\sum_{k=0}^{\infty} F_k x^k = x + \sum_{k=2}^{\infty} F_k x^k = x + \sum_{k=2}^{\infty} (F_{k-1} + F_{k-2}) x^k = x + \sum_{k=2}^{\infty} F_{k-1} x^k + \sum_{k=2}^{\infty} F_{k-2} x^k$$

Now we have the equation

$$\sum_{k=0}^{\infty} F_k x^k = x + \sum_{k=2}^{\infty} F_{k-1} x^k + \sum_{k=2}^{\infty} F_{k-2} x^k$$

and our goal is to modify the right-hand side so the only sums we see are $\sum_{k=0}^{\infty} F_k x^k$.

Express $\sum_{k=2}^{\infty} F_{k-1} x^k$ and $\sum_{k=2}^{\infty} F_{k-2} x^k$ in terms of $\sum_{k=0}^{\infty} F_k x^k$. *Hint:* Write out the first few terms of each of the series. Your main tools will be factoring and subtracting.

Using what you got in the previous problem, plug into the equation

$$\sum_{k=0}^{\infty} F_k x^k = x + \sum_{k=2}^{\infty} F_{k-1} x^k + \sum_{k=2}^{\infty} F_{k-2} x^k$$

and show that

$$F(x) = \sum_{k=0}^{\infty} F_k x^k = \frac{x}{1 - x - x^2}$$

5 Obtaining A Power Series

Now we know that

$$F(x) = \frac{x}{1 - x - x^2}$$

Now, we can use *partial fractions* to separate $F(x)$ into more manageable pieces.

Show that

$$\frac{x}{1 - x - x^2} = \frac{1}{\sqrt{5}}\left(\frac{1}{1 - \varphi x}\right) - \frac{1}{\sqrt{5}}\left(\frac{1}{1 - \varphi' x}\right)$$

where

$$\varphi = \frac{1 + \sqrt{5}}{2}$$
$$\varphi' = \frac{1 - \sqrt{5}}{2}$$

6 Bringing It All Together

Using the results from this workshop, show how Binet's Formula is obtained.