Exam 1

This exam covers topics from Sections 6.1-6.3,6.5, 7.1-7.5,7.7,7.8, 9.1, and 9.2 in the Stewart book. Unless stated explicitly in the problem, **you must show all work for full credit**. Put a box around your final answers. This exam is meant to be taken over 80 minutes, but 100 minutes will be given for the exam. The total for the exam is 100 points. It counts for 20 percent of your final grade.

By signing below, you promise that you will not cheat in any way on this exam.

"The ball is round, the game lasts ninety minutes, and everything else is just theory."

Former German soccer coach Sepp Herberger.

Name:

Signature:

Problem	**Points Possible**	**Points Earned**
1a	8	
1b	8	
1c	8	
1d	8	
1e	8	
2a	10	
2b	5	
2c	5	
3a	10	
3b	10	
4a	10	
4b	10	
Grade:		

1. **Integration Time.** Compute the following integrals. Use whatever method you like.
(a)

$$\int \ln^2 x \; dx$$

We let

$$\begin{array}{ll} u = \ln^2 x & v = x \\ du = \frac{2x}{\ln x}\; dx & dv = 1\; dx \end{array}$$

and so

$$\int \ln^2 x \; dx = x \ln^2 x - 2 \int \ln x \; dx$$

and again

$$\begin{array}{ll} u = \ln x & v = x \\ du = \frac{1}{x}\; dx & dv = 1\; dx \end{array}$$

and so

$$\int \ln^2 x \; dx = x \ln^2 x - 2(x \ln x - x) = x \ln^2 x - 2x \ln x + 2x$$

(b)

$$\int \frac{3x^2 - 4x + 5}{(x-1)^2(x^2+x+1)}\; dx$$

We set up partial fractions:

$$\frac{3x^2 - 4x + 6}{(x-1)^2(x^2+x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx + D}{x^2+x+1}$$

which yields the equations

$$\begin{array}{rcl} 0 & = & A + C \\ 3 & = & B - 2C + D \\ -4 & = & B + C - 3D \\ 5 & = & -a + B + D \end{array}$$

with solutions

$$\begin{aligned} A &= -2/3 \\ B &= 4/3 \\ C &= 2/3 \\ D &= 3 \end{aligned}$$

So we have

$$\begin{aligned} \int \frac{3x^2-4x+5}{(x-1)^2(x^2+x+1)} &= -\frac{2}{3}\int \frac{1}{x-1}\,dx + \frac{4}{3}\int \frac{1}{(x-1)^2}\,dx + \int \frac{\frac{2}{3}x+3}{x^2+x+1}\,dx \\ &= -\frac{2}{3}\int \frac{1}{x-1}\,dx + \frac{4}{3}\int \frac{1}{(x-1)^2}\,dx + \int \frac{\frac{2}{3}x+\frac{1}{3}}{x^2+x+1}\,dx + \frac{8}{3}\int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx \\ &= -\frac{2}{3}\ln|x-1| - \frac{4/3}{x-1} + \frac{1}{3}\ln|x^2+x+1| + \frac{2}{\sqrt{3}}\tan^{-1}(\frac{2}{\sqrt{3}}x - \frac{1}{\sqrt{3}}) \end{aligned}$$

(c)

$$\int \sin^3 x \cos^5 x \; dx$$

We have

$$\begin{aligned}
\int \sin^3 x \cos^5 x \; dx &= \int \sin x (1 - \cos^2 x) \cos^5 x \; dx \\
&= \int \sin x \cos^5 x \; dx - \int \sin x \cos^7 x \; dx \\
&= \frac{\cos^6 x}{6} - \frac{\cos^8 x}{8}
\end{aligned}$$

(d)

$$\int \frac{1}{\sqrt{9t^2 - 16}} \; dt$$

If we let $u = 3t$ then we have

$$\int \frac{1}{\sqrt{9t^2 - 16}} \; dt = \frac{1}{3} \int \frac{1}{\sqrt{u^2 - 16}} \; du = \frac{1}{3} \ln |u + \sqrt{u^2 - 16}| = \frac{1}{3} \ln |3t - \sqrt{9t^2 - 16}|$$

where the last computation came from the table of integrals in the back.

(e)

$$\int \frac{1}{e^{3x} - e^x}\, dx$$

(Hint: let $y = e^x$) If we let

$$\begin{aligned} y &= e^x \\ dy &= e^x\, dx = y\, dx \end{aligned}$$

Then we have

$$\int \frac{1}{e^{3x} - e^x}\, dx = \int \frac{1}{y(y^3 - y)}\, dy = \int \frac{1}{y^2(y+1)(y-1)}\, dy$$

which is just a partial fraction problem. So we let

$$\frac{1}{y^2(y+1)(y-1)} = \frac{A}{y} + \frac{B}{y^2} + \frac{C}{y+1} + \frac{D}{y-1}$$

which has solutions

$$\begin{aligned} A &= 0 \\ B &= -1 \\ C &= -1/2 \\ D &= 1/2 \end{aligned}$$

So we end up with

$$\int \frac{1}{y^2(y+1)(y-1)}\, dy = \frac{1}{y} - \frac{1}{2}\ln|y+1| + \frac{1}{2}\ln|y-1| = \frac{1}{e^x} - \frac{1}{2}\ln|e^x+1| + \frac{1}{2}\ln|e^x-1|$$

2. **A lesson on probability.** We mathematicians say that X is a *continuous random variable* if there exists a non-negative function f such that

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

With this, we can define *the probability that X is greater than a*, written as $Pr(X > a)$, as

$$Pr(X > a) = \int_{a}^{\infty} f(x)\, dx$$

Similarly, we can define *the probability that X is less than a*, written as $Pr(X < a)$, as

$$Pr(X < a) = \int_{-\infty}^{b} f(x)\, dx$$

Finally, we can define *the probability that X is between a and b*, written as $Pr(a < X < b)$, as

$$Pr(a < X < b) = \int_{a}^{b} f(x)\, dx$$

Given $\lambda > 0$, we define a continuous random variable X, called the *exponential random variable with parameter λ* by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

(a) Show that our exponential random variable is indeed a continuous random variable by showing

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

Well,

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} \lambda e^{-\lambda x}\, dx = \int_{0}^{\infty} \lambda e^{-\lambda x}\, dx = (-e^{-\lambda x})|_0^{\infty} = 1$$

(b) If a satisfies $Pr(X > a) = .5$, what is a?

We solve:

$$\int_a^{\infty} \lambda e^{-\lambda x}\, dx = (-e^{-\lambda x})|_a^{\infty} = e^{-\lambda a} = .5$$

So

$$a = -\frac{\ln .5}{\lambda}$$

(c) What is $Pr(-5 < X < 10)$?

We compute:

$$\int_{-5}^{10} \lambda e^{-\lambda x}\, dx = \int_0^{10} \lambda e^{-\lambda x}\, dx = -e^{-\lambda x}|_0^1 0 = 1 - e^{-10\lambda}$$

3. **Volumes.** Let A be the area bounded by the curves

$$\begin{aligned} y &= x^3 + 6x^2 + 11x + 6 \\ x &= 0 \\ y &= 9 \end{aligned}$$

(a) Find the volume of the solid obtained by rotating A around the x**-axis**.

We will use slicing - at some x-value, the cross-section is a disc with outer radius 24 and inner radius $x^3 + 6x^2 + 11x + 6$ so the area of the disc is $\pi(24)^2 - \pi(x^3 + 6x^2 + 11x + 6)^2$. So the volume is

$$V = \int_0^1 \pi(24)^2 - \pi(x^3 + 6x^2 + 11x + 6)^2 \, dx$$

(b) Find the volume of the solid obtained by rotating A around the y**-axis**.

We will use cylindrical cross-sections. At some x-value, the cross-section is a cylinder with radius x and height $24 - (x^3 + 6x^2 + 11x + 6)$. Hence the area of this cross-section is $2\pi x(24 - (x^3 + 6x^2 + 11x + 6))$ so the volume is

$$\int_0^1 2\pi x(24 - (x^3 + 6x^2 + 11x + 6)) \, dx$$

4. **A lesson on differential equations.** Consider the differential equation

$$y'' - 9y' + 20y = 0$$

The best way to solve a differential equation is to guess, and since I am a pretty good guesser, I am going to guess that

$$y = e^{rx}$$

is a solution for some real number r.

(a) There are two values of r such that e^{rx} is a solution to the differential equation above. What are the two values?

Well,

$$\begin{aligned} y &= e^{rx} \\ y' &= re^{rx} \\ y'' &= r^2 e^{rx} \end{aligned}$$

and so we get

$$y'' - 9y' + 20y = r^2 e^{rx} - 9re^{rx} + 20e^{rx} = e^{rx}(r^2 - 9r + 20) = 0$$

So we need to solve

$$r^2 - 9r + 20 = 0$$

which has solutions $r = 4, 5$.

(b) If r_1 and r_2 are the values found in part (a), show that for any constants A and B,

$$Ae^{r_1 x} + Be^{r_2 x}$$

is a solution to the differential equation above.

Well, just as in part (a), we have:

$$\begin{aligned} y &= Ae^{r_1 x} + Be^{r_2 x} \\ y' &= Ar_1 e^{r_1 x} + Br_2 e^{r_2 x} \\ y'' &= Ar_1^2 e^{r_1 x} + Br_2^2 e^{r_2 x} \end{aligned}$$

So

$$\begin{aligned}
y'' - 9y' + 20y &= Ar_1^2 e^{r_1 x} + Br_2^2 e^{r_2 x} - 9(Ar_1 e^{r_1 x} + Br_2 e^{r_2 x}) + 20(Ae^{r_1 x} + Be^{r_2 x}) \\
&= A(r_1^2 e^{r_1 x} - 9r_1 e^{r_1 x} + 20e^{r_1 x}) + B(r_2^2 e^{r_2 x} - 9e^{r_2 x} + 20e^{r_2 x}) \\
&= A(y'' - 9y' + 20y) + B(y'' - 9y' + 20y) \\
&= 0
\end{aligned}$$