Exam 2

This exam covers topics from sections 9.3 and 11.1-11.9 in the Stewart book. Unless stated explicitly in the problem, **you must show all work for full credit**. Put a box around your final answers. This exam is meant to be taken over 80 minutes, but 100 minutes will be given for the exam. The total for the exam is 100 points. It counts for 20 percent of your final grade.

By signing below, you promise that you will not cheat in any way on this exam.

"The ball is round, the game lasts ninety minutes, and everything else is just theory."

Former German soccer coach Sepp Herberger.

Name:

Signature:

Problem	**Points Possible**	**Points Earned**
1a	10	
1b	10	
2a	8	
2b	8	
2c	8	
2d	8	
2e	8	
3	15	
4a	5	
4b	10	
4c	10	
Grade:		

1. Examples.

(a) Give an example of a sequence, $\{a_n\}$, that is bounded, not monotonic, and divergent.

One example (of many) would be $1, -1, 1, -1, 1, -1, 1, -1, \cdots$, or

$$a_n = (-1)^{n-1}$$

(b) Give an example of two sequences $\{a_n\}$ and $\{b_n\}$ such that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are divergent, but

$$\sum_{n=1}^{\infty} a_n b_n$$

is convergent.

One example (of many) would be

$$a_n = b_n = \frac{1}{n}$$

2. A world of series. For each of the following five series, circle AC if the series is absolutely convergent, CC if the series is conditionally convergent, or D if the series is divergent. Use whatever method you would like. Only circle one of the three per row! **Note: There will be no partial credit on this problem - each one is all or nothing.**

(a)	$\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n-5}$	D
(b)	$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n \ln n (\ln \ln n)^{100}}$	D
(c)	$\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$	CC
(d)	$\displaystyle\sum_{n=1}^{\infty} \frac{3^n + 4^n}{7^n}$	AC
(e)	$\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)\left(\frac{n+3}{n+2}\right)$	D

Hint: $u = \ln \ln x$ would help somewhere.

3. Power to the power series.

As you will learn very soon, the power series representation for $\cos x$ is

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

For what values of x does this power series converge?

We can use the ratio test. We have

$$\begin{aligned} a_n &= (-1)^n \frac{x^{2n}}{(2n)!} \\ a_{n+1} &= (-1)^{n+1} \frac{x^{2(n+1)}}{(2(n+1))!} \\ \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{x^2}{(2n+2)(2n+1)} \right| \end{aligned}$$

Note that, for any x,

$$\lim_{n\to\infty} \left| \frac{x^2}{(2n+2)(2n+1)} \right| = |x^2| \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0$$

So this series converges everywhere.

4. Exponential Generating Functions.

Consider the sequence $\{a_n\}$ defined recursively, by

$$\begin{aligned} a_0 &= 1 \\ a_1 &= 1 \\ a_n &= a_{n-1} + (n-1)a_{n-2} \end{aligned}$$

(a) Write the next 5 numbers in this sequence.

$$\begin{aligned} a_2 &= a_1 + 1 \cdot a_0 = 1 + 1 = 2 \\ a_3 &= a_2 + 2a_1 = 2 + 2(1) = 4 \\ a_4 &= a_3 + 3a_2 = 4 + 3(2) = 10 \\ a_5 &= a_4 + 4a_3 = 10 + 4(4) = 26 \\ a_6 &= a_5 + 5a_4 = 26 + 5(10) = 76 \end{aligned}$$

Now, our goal is to find the function $f(x)$ whose power series is

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}$$

We will do so as follows:

(b) Show that $f(x)$ satisfies

$$f'(x) = f(x) + xf(x)$$

I will start it out for you (make sure you understand the steps, especially the second one):

$$\begin{aligned} f'(x) &= \sum_{n=0}^{\infty} \frac{n a_n x^{n-1}}{n!} \\ &= \sum_{n=0}^{\infty} a_{n+1} \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} (a_n + n a_{n-1}) \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} + \sum_{n=0}^{\infty} a_{n-1} \frac{x^n}{(n-1)!} \\ &= \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} + x \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \\ &= f(x) + xf(x) \end{aligned}$$

Notice that this is a differential equation - if we let $y = f(x)$, then this is

$$y' = y + xy$$

(c) Solve this differential equation for y, and we're done!
Note: Even if you can't do part (b), you should still do this part!

We can separate:

$$\begin{aligned}
\frac{dy}{dx} &= y + xy \\
\frac{1}{y}\, dy &= 1 + x\, dx \\
\int \frac{1}{1+y}\, dy &= \int 1 + x\, dx \\
\ln y &= x + x^2 \\
y &= e^{x+x^2}
\end{aligned}$$

For those who care: The numbers a_n described in this problems are the number of *involutions* of the first n numbers.