18 July 2006

Solutions - 10.2

3. We have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4t^3}{3t^2}$$

So at $t = -1$ we have

$$\frac{dy}{dx} = -\frac{4}{3}$$

6. We have

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos\theta + 2\sin 2\theta}{-\sin\theta + 2\cos 2\theta}$$

So we $\theta = 0$ we have

$$\frac{dy}{dx} = \frac{1+0}{-0+2} = \frac{1}{2}$$

9. A t-value this corresponds to is $t = \frac{\pi}{3}$. So we have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\cos t}{-2\sin t} = -\cot t$$

and so at $t = \frac{\pi}{3}$ we have

$$\frac{dy}{dx} = -\frac{1}{\sqrt{3}}$$

So our equation will be

$$y - 1 = -\frac{1}{\sqrt{3}}(x - \sqrt{3})$$

17. Done in class . . .

25. Well, the fact that there are two tangents comes from the fact that we are at $(0,0)$ when $t = \frac{\pi}{2}$ and when $t = -\frac{\pi}{2}$. We have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos^2 t - \sin^2 t}{-\sin t}$$

So at $t = \frac{\pi}{2}$ we have

$$\frac{dy}{dx}(\frac{\pi}{2}) = 1$$

so we are vertical. At $t = -\frac{\pi}{2}$ we have

$$\frac{dy}{dx}(-\frac{\pi}{2}) = -1$$

26. Well, if we let t and t' be the two time points we are looking for, then we need to solve

$$\begin{aligned} 1 - 2\cos^2 t &= 1 - 2\cos^2 t' \\ (\tan t)(1 - 2\cos^2 t) &= (\tan t')(1 - 2\cos^2 t') \end{aligned}$$

We can see that these are both satisfied when $t = \frac{\pi}{4}$ and $t = -\frac{\pi}{4}$. We can now compute the derivative:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sec^2 t(1 - 2\cos^2 t) + \tan t 4\cos t \sin t}{4\cos t \sin t}$$

So at $t = \frac{\pi}{4}$ we have

$$\frac{dy}{dx}(\frac{\pi}{4}) = 1$$

whereas at $t = -\frac{\pi}{4}$ we have

$$\frac{dy}{dx}(-\frac{\pi}{4}) = -1$$

39. The answer is

$$\int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{(1 - \sin t)^2 + (1 - \cos t)^2}\, dt$$

41. We need to evaluate

$$\int_0^1 \sqrt{(6t)^2 + (6t)^2}\, dt = \sqrt{72} \int_0^1 t\, dt = 3\sqrt{2}$$

44. We need to evaluate

$$\int_0^3 \sqrt{(e^t - e^{-t})^2 + (-2)^2}\, dt = \int_0^3 \sqrt{e^{2t} + e^{-2t} + 2}\, dt = \int_0^3 e^{2t} + 1\, dt$$

continuing:

$$\int_0^3 e^{2t} + 1\, dt = (\frac{e^{2t}}{2} + t)|_0^3 = (\frac{e^6}{2} + 3) - (\frac{1}{2}) = \frac{e^6 - 1}{2} + 3$$

52. Notice first that the distance covered by the particle is going to be 4 times the length of the curve, which is

$$\int_0^{\pi} \sqrt{4\cos^2 t \sin^2 t + \sin^2 t}\, dt = \int_0^{\pi} \sin t\sqrt{4\cos^2 t + 1}\, dt = \left(-\frac{2}{12}(4\cos t + 1)^{\frac{3}{2}} \right)\Big|_0^{\pi}$$

This evaluates to

$$\frac{1}{6}(5^{3/2} - 3^{3/2})$$

53. Well, notice that the length of the ellipse is 4 times the length of the first quadrant, which is

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\, d\theta &= a\int_0^{\frac{\pi}{2}} \sqrt{\cos^2\theta + \frac{b^2}{a^2}\sin^2\theta\, d\theta} \\
&= a\int_0^{\frac{\pi}{2}} \sqrt{(\cos^2\theta + \sin^2\theta) + (\frac{b^2}{a^2} - 1)\sin^2\theta\, d\theta} \\
&= a\int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{a^2 - b^2}{a^2}\sin^2\theta\, d\theta} \\
&= a\int_0^{\frac{\pi}{2}} \sqrt{1 - e^2\sin^2\theta}\, d\theta
\end{aligned}$$

and we are done.

57. We have

$$\int_1^2 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_1^2 2\pi \left(\frac{4}{3}t^{\frac{3}{2}}\right)\sqrt{(1-2t)^2 + t}\, dt$$

61. We have

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, d\theta &= 2\pi a\int_0^{\frac{\pi}{2}} \sin^3\theta\sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\, d\theta \\
&= 6\pi a^2\int_0^{\frac{\pi}{2}} \sin^4\theta\cos\theta\, d\theta \\
&= 6\pi a^2\left(\frac{\sin^5\theta}{5}\right)\Big|_0^{\frac{\pi}{2}} \\
&= \frac{6\pi a^2}{5}
\end{aligned}$$