Solutions: 6.2

Sketches will not be given. Only math is presented here, although you should make sure to draw the sketches yourself! Also, I do not guarantee correctness! The answers may not be correct, but the path to the answers are.

3. A cross-section at some x-value will be a circle with radius $1/x$ and hence area $\frac{\pi}{x^2}$. Hence, the integral we need to compute is

$$\int_1^2 \frac{\pi}{x^2} dx = -\frac{\pi}{x}\Big|_1^2 = -\frac{\pi}{2} - -\frac{\pi}{1} = \pi = \frac{\pi}{2} = \frac{\pi}{2}$$

6. A cross-section at some y-value will be a circle with radius $y - y^2$ and hence area $\pi(y-y^2)^2 = \pi(y^2 - 2y^3 + y^4)$. For the top and bottom boundaries of the solid (for the limits of integration), we need to solve

$$y - y^2 = 0$$

which gives us $y = 0$ and $y = 1$. Hence the integral we need to compute is

$$\begin{aligned} \int_0^1 \pi(y^2 - 2y^3 + y^4)dy &= \pi \int_0^1 (y^2 - 2y^3 + y^4)dy \\ &= \pi \left(\frac{y^3}{3} - \frac{y^4}{2} + \frac{y^5}{5}\right)\Big|_0^1 \\ &= \pi \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) - \pi(0) \\ &= \frac{\pi}{30} \end{aligned}$$

7. A cross-section at some x-value will be a washer with outer radius $\sqrt{x}$ and inner radius x^2. Therefore, the area of this washer is $\pi(\sqrt{x})^2 - \pi(x^2)^2 = \pi(x - x^4)$. The limits of integration are the intersection points, which are 0 and 1. Therefore, the integral we need to compute is

$$\begin{aligned} \int_0^1 \pi(x - x^4)dx &= \pi \int_0^1 (x - x^4)dx \\ &= \pi \left(\frac{x^2}{2} - \frac{x^5}{5}\right)\Big|_0^1 \\ &= \pi \left(\frac{1}{2} - \frac{1}{5}\right) \\ &= \frac{3\pi}{10} \end{aligned}$$

9. A cross-section at some y-value will be a washer with outer radius $2y$ and an inner radius of $\sqrt{y}$. Therefore, the area of this washer is $\pi(2y)^2 - \pi(\sqrt{y})^2 = \pi(4y^2 - y)$. The limits of integration are the intersection points, which are the solutions to the equation

$$y^2 = 2y$$

which is $y = 0$ and $y = 2$. hence, the integral we need to compute is

$$\begin{aligned}\int_0^2 \pi(4y^2 - y)dy &= \pi \int_0^2 (4y^2 - y)dy \\ &= \pi\left(\frac{4y^3}{3} - \frac{y^2}{2}\right)\Bigg|_0^2 \\ &= \pi\left(\frac{32}{3} - 2\right) \\ &= \frac{26\pi}{3}\end{aligned}$$

12. A cross-section at some x-value will be a circle with radius $4 - x^2$. The area of the circle is $\pi(4 - x^2)^2 = \pi(16 - 8x^2 + x^4)$. The limits of integration are the points of intersection, which is $x = -2$ and $x = 2$. Hence the integral we need to compute is

$$\begin{aligned}\int_{-2}^2 \pi(16 - 8x^2 + x^4)dx &= \pi \int_{-2}^2 \pi(16 - 8x^2 + x^4)dx \\ &= \pi\left(16x - \frac{8x^3}{3} + \frac{x^5}{5}\right)\Bigg|_{-2}^2 \\ &= 2\pi\left(32 - \frac{64}{3} + \frac{32}{5}\right) \qquad \text{(why?)} \\ &= \frac{512\pi}{15}\end{aligned}$$

17. A cross-section at some y-value will be a washer with outer radius $1+\sqrt{y}$ and inner radius $1+y^2$. The area of the washer is therefore $\pi(1+\sqrt{y})^2 - \pi(1+y^2)^2 = \pi((1+2\sqrt{y}+y) - (1+2y^2+y^4)) = \pi(2\sqrt{y}+y-2y^2-y^4)$. The limits of integration are the points of intersection, which occurs at $y = 0$ and $y = 1$. Hence the integral we need to compute is

$$\begin{aligned}\int_0^1 \pi(2\sqrt{y} + y - 2y^2 - y^4)dy &= \pi \int_0^1 (2\sqrt{y} + y - 2y^2 - y^4)dy \\ &= \pi\left(\frac{2y^{1.5}}{1.5} + \frac{y^2}{2} - \frac{2y^3}{3} - \frac{y^5}{5}\right)\Bigg|_0^1 \\ &= \pi\left(\frac{4}{3} + \frac{1}{2} - \frac{2}{3} - \frac{1}{5}\right) \\ &= \frac{29\pi}{30}\end{aligned}$$

26. A cross-section at some x-value will be a circle with radius $1 - \sqrt{x}$ and area $\pi(1-\sqrt{x})^2 = \pi(1-2\sqrt{x}+x)$. Hence, the integral we need to compute is

$$\begin{aligned}\int_0^1 \pi(1 - 2\sqrt{x} + x)dx &= \pi \int_0^1 (1 - 2\sqrt{x} + x)dx \\ &= \pi\left(x - \frac{2x^{1.5}}{1.5} + \frac{x^2}{2}\right)\Bigg|_0^1 \\ &= \pi\left(1 - \frac{4}{3} + \frac{1}{2}\right) \\ &= \frac{\pi}{6}\end{aligned}$$

28. A cross-section at some y-value will be a washer with outer radius of $\sqrt[3]{y}$ and an inner radius of y^2. Hence the area of the washer will be $\pi(y^{1.5} - y^4)$. Therefore, the integral we need to compute is

$$\begin{aligned}
\int_0^1 \pi(y^{1.5} - y^4)dy &= \pi \int_0^1 (y^{1.5} - y^4)dy \\
&= \pi \left(\frac{y^{2.5}}{2.5} - \frac{y^5}{5} \right)\Big|_0^1 \\
&= \pi \left(\frac{2}{5} - \frac{1}{5} \right) \\
&= \frac{\pi}{5}
\end{aligned}$$

54. If you imagine the solid S as standing on its circular base, we will integrate from left to right, i.e. from $-r$ to r. A cross-section is a square whose side length is $2\sqrt{r^2 - x^2}$ (why?). Hence the integral we need to calculate is

$$\begin{aligned}
\int_{-r}^{r} 4(r^2 - x^2)dx &= 4r^2 \int_{-r}^{r} 1dx - 4\int_{-r}^{r} x^2 dx \\
&= 4\, r^2(x)\big|_{-r}^{r} - 4\left(\frac{x^3}{3}\right)\Big|_{-r}^{r} \\
&= 8r^3 - \frac{8r^3}{3} \\
&= \frac{16r^3}{3}
\end{aligned}$$

55. The two points of the ellipse having some specific x-value are $(x, \pm\sqrt{\frac{36-9x^2}{4}})$ and so the length of the hypotenuse at some x-value will be $2\sqrt{\frac{36-9x^2}{4}}$. Therefore the area of the right triangle will be $\frac{36-9x^2}{2}$. The ellipse is defined only between $-2 \le x \le 2$ and so the integral we need to compute is

$$\begin{aligned}
\int_{-2}^{2} \frac{36 - 9x^2}{2} dx &= \frac{1}{2}\int_{-2}^{2} (36 - 9x^2)dx \\
&= \frac{1}{2}\left(36x - 3x^3\right)\big|_{-2}^{2} \\
&= \frac{1}{2}((72 - 24) - (-72 + 24)) \\
&= \frac{1}{2}(96) \\
&= 48
\end{aligned}$$