Solutions: 6.3

1. It is awkward to use slicing since it would be a true pain to figure out the inner and outer radii for each y-value. It would get to a point where the integral is simply not solvable. To use shells, let some x-value be given. Then the area of the cylindrical shell will be

$$2\pi x(x)(x-1)^2$$

So our volume would be

$$\begin{aligned}\int_0^1 2\pi x^2(x-1)^2\,dx &= 2\pi\int_0^1 x^4-2x^3+x^2\,dx\\ &= 2\pi\left(\frac{x^5}{5}-\frac{x^4}{2}+\frac{x^3}{3}\right)\Big|_0^1\\ &= \frac{\pi}{15}\end{aligned}$$

4. At some x-value, our cylindrical cross-section has radius x and height x^2, so the total area of $2\pi x\cdot x^2 = 2\pi x^3$. Therefore, our volume is

$$\begin{aligned}\int_0^1 2\pi x^3\,dx &= 2\pi\left(\frac{x^4}{4}\right)\Big|_0^1\\ &= \frac{\pi}{2}\end{aligned}$$

39. First, we need to find the points of intersection, meaning we need to solve

$$\begin{aligned}x+\frac{4}{x} &= 5\\ x^2-5x+4 &= 0\\ (x-4)(x-1) &= 0\\ (4,5) \quad &\text{and} \quad x=(1,5)\end{aligned}$$

So at some x-value, $1\le x\le 4$, our cylindrical shell has radius $1+x$ and height $5-(x+\frac{4}{x})$, and therefore has area $2\pi(1+x)\left(5-x-\frac{4}{x}\right)$. So we need to compute

$$\begin{aligned}\int_1^4 2\pi(1+x)\left(5-x-\frac{4}{x}\right)\,dx &= 2\pi\int_1^4(1-\frac{4}{x}+4x-x^2)\,dx\\ &= 2\pi\left(x-4\ln x+2x^2-\frac{x^3}{3}\right)\Big|_1^4\\ &= 2\pi(4-4\ln 4+32-\frac{64}{3}-(1-4\ln 1+2-\frac{1}{3}))\\ &= \pi(6-2\ln 4)\end{aligned}$$

40. In this case, we shall use slicing. For some y-value, $0\le y\le 1$ (why those bounds?) the cross-section is a washer with outer radius 2 and inner radius $2-(1-y^4)=1+y^4$. Hence the total area is $\pi(2)^2-\pi(1+y^4)^2=\pi(y^8+2y^4+5)$. Hence the volume that we seek is

$$\begin{aligned}\int_{-1}^1 \pi(y^8+2y^4+5)\,dy &= \pi\int_{-1}^1 y^8-2y^4+5\,dy\\ &= \pi\left(\frac{y^9}{9}-\frac{2y^5}{5}+5y\right)\Big|_{-1}^1\\ &= \frac{424\pi}{45}\end{aligned}$$

45. We shall use cylindrical shells. Consider some cross section that is x units away from the center of the base, so $0 \leq x \leq r$. If we let h_x be the height of the cylindrical cross-section at x, then by similar triangles we have

$$\frac{h}{r} = \frac{h_x}{r-x}$$

So $h_x = \frac{h(r-x)}{r}$. Therefore, the area of the cross-section is $2\pi x \frac{h(r-x)}{r}$. So the integral we need is

$$\begin{aligned}
\int_0^r 2\pi x \frac{h(r-x)}{r}\, dx &= \frac{2\pi h}{r} \int_0^r rx - x^2\, dx \\
&= \frac{2\pi h}{r} \left(\frac{rx^2}{2} - \frac{x^3}{3} \right) \Bigg|_0^r \\
&= \frac{2\pi h}{r} \left(\frac{r^3}{2} - \frac{r^3}{3} \right) \\
&= \frac{2\pi h}{r} \cdot \frac{r^3}{6} \\
&= \frac{1}{3}\pi r^2 h
\end{aligned}$$

Which is indeed what it is supposed to be!