Solutions: 7.1

3. We choose

$$\begin{array}{ll} u = x & v = \frac{1}{5}\sin 5x \\ du = 1\ dx & dv = \cos 5x\ dx \end{array}$$

So we get

$$\begin{aligned} \int x\cos 5x\ dx &= \frac{1}{5}x\sin 5x - \int \frac{1}{5}\sin 5x\ dx \\ &= \frac{x\sin 5x}{5} + \frac{1}{25}\cos 5x + C \end{aligned}$$

8. We choose

$$\begin{array}{ll} u = x^2 & v = \frac{1}{m}\sin mx \\ du = 2x\ dx & dv = \cos mx\ dx \end{array}$$

We obtain

$$\begin{aligned} \int x^2\cos mx\ dx &= \frac{1}{m}x^2\sin mx - \int \frac{2x}{m}\sin mx\ dx \\ &= \frac{x^2\sin mx}{m} - \frac{2}{m}\int x\sin mx\ dx \end{aligned}$$

Now we need to evaluate $\int x\sin mx\ dx$. So now we choose

$$\begin{array}{ll} u = x & v = -\frac{1}{m}\cos mx \\ du = 1\ dx & dv = \sin mx\ dx \end{array}$$

and we get

$$\begin{aligned} \int x\sin mx\ dx &= -\frac{x\cos mx}{m} + \int \frac{1}{m}\cos mx\ dx \\ &= -\frac{x\cos mx}{m} + \frac{\sin mx}{m^2} \end{aligned}$$

So combining the two we obtain

$$\begin{aligned} \int x^2\cos mx &= \frac{x^2\sin mx}{m} - \frac{2}{m}\left(-\frac{x\cos mx}{m} + \frac{\sin mx}{m^2}\right) \\ &= \frac{x^2\sin mx}{m} + \frac{2x\cos mx}{m^2} - \frac{2\sin mx}{m^3} + C \end{aligned}$$

10. We choose

$$\begin{array}{ll} u = \sin^{-1} x & v = x \\ du = \frac{1}{\sqrt{1-x^2}}\ dx & dv = 1\ dx \end{array}$$

and we get

$$\begin{aligned} \int \sin^{-1} x\ dx &= x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\ dx \\ &= x\sin^{-1} x + \sqrt{1-x^2} + C \end{aligned}$$

13. We choose

$$\begin{array}{ll} u = (\ln x)^2 & v = x \\ du = \frac{2\ln x}{x}\ dx & dv = 1\ dx \end{array}$$

and we obtain

$$\begin{aligned}\int (\ln x)^2 \, dx &= x(\ln x)^2 - \int 2\ln x \, dx \\ &= x(\ln x)^2 - 2(x\ln x - x) + C\end{aligned}$$

15. We choose

$$\begin{array}{ll} u = \sin 3\theta & v = \frac{1}{2}e^{2\theta} \\ du = 3\cos 3\theta d\theta & dv = e^{2\theta} d\theta \end{array}$$

and we obtain

$$\int e^{2\theta}\sin 3\theta d\theta = \frac{1}{2}\sin 3\theta e^{2\theta} - \frac{3}{2}\int e^{2\theta}\cos 3\theta d\theta$$

and now we must integrate $\int e^{2\theta}\cos 3\theta$, and now we choose

$$\begin{array}{ll} u = \cos 3\theta & v = \frac{1}{2}e^{2\theta} \\ du = -3\sin 3\theta d\theta & dv = e^{2\theta} d\theta \end{array}$$

and now we have

$$\begin{aligned}\frac{1}{2}\sin 3\theta e^{2\theta} - \frac{3}{2}\int e^{2\theta}\cos 3\theta d\theta &= \frac{1}{2}\sin 3\theta e^{2\theta} - \frac{3}{2}\left(\frac{1}{2}\cos^{3\theta} e^{2\theta} + \frac{3}{2}\int e^{2\theta}\sin 3\theta d\theta\right) \\ &= \frac{1}{2}\sin 3\theta e^{2\theta} - \frac{3}{4}\cos 3\theta e^{2\theta} - \frac{9}{4}\int e^{2\theta}\sin 3\theta d\theta\end{aligned}$$

So we get the equation

$$\frac{13}{4}\int e^{2\theta}\sin 3\theta d\theta = \frac{1}{2}\sin 3\theta e^{2\theta} - \frac{3}{4}\cos 3\theta e^{2\theta}$$

and so

$$\int e^{2\theta}\sin 3\theta d\theta = \frac{2}{13}\sin 3\theta e^{2\theta} - \frac{3}{13}\cos 3\theta e^{2}\theta + C$$

20. We choose

$$\begin{array}{ll} u = x^2 + 1 & v = -e^{-x} \\ du = 2x \, dx & dv = e^{-x} \, dx \end{array}$$

and so

$$\int (x^2+1)e^{-x} \, dx = -(x^2+1)e^{-x} + 2\int xe^{-x} \, dx$$

and for this new integral we choose

$$\begin{array}{ll} u = x & v = -e^{-x} \\ du = 1 \, dx & dv = e^{-x} \, dx \end{array}$$

and we get

$$\begin{aligned}\int (x^2+1)e^{-x} \, dx = -(x^2+1)e^{-x} + 2\int xe^{-x} \, dx &= -(x^2+1)e^{-x} + 2(-xe^{-x} + \int e^{-x} \, dx) \\ &= -(x^2+1)e^{-x} - 2xe^{-x} - 2e^{-x} + C\end{aligned}$$

And so

$$\begin{aligned}\int_0^1 (x^2+1)e^{-x} &= (-(x^2+1)e^{-x} - 2xe^{-x} - 2e^{-x} + C)|_0^1 \\ &= (-2e^{-1} - 2e^{-1} - 2e^{-1}) - (-1 - 0 - 2) \\ &= \frac{-6}{e} + 3\end{aligned}$$

21. We choose

$$\begin{array}{ll} u = \ln x & v = -\frac{1}{x} \\ du = \frac{1}{x}\, dx & dv = \frac{1}{x^2}\, dx \end{array}$$

and we obtain

$$\begin{aligned} \int \frac{\ln x}{x^2}\, dx &= -\frac{\ln x}{x} + \int \frac{1}{x^2}\, dx \\ &= -\frac{\ln x}{x} - \frac{1}{x} + C \end{aligned}$$

so

$$\int_1^2 \frac{lnx}{x^2}\, dx = \left(-\frac{\ln x}{x} - \frac{1}{x}\right)\bigg|_1^2 = \left(-\frac{\ln 2}{2} - \frac{1}{2}\right) - (0-1) = \frac{1-\ln 2}{2}$$

41. (a) We choose

$$\begin{array}{ll} u = \sin x & v = -\cos x \\ du = \cos x\, dx & dv = \sin x\, dx \end{array}$$

and we get

$$\int \sin^2 x\, dx = -\sin x \cos x + \int \cos^2 x\, dx = -\frac{\sin 2x}{2} + \int 1\, dx - \int \sin^2 x\, dx$$

and so solving for $\int \sin^2 x\, dx$ we obtain

$$\int \sin^2 x\, dx = -\frac{\sin 2x}{4} + \frac{x}{2} + C$$

(b) We choose

$$\begin{array}{ll} u = \sin^2 x & v = -\frac{\sin 2x}{4} + \frac{x}{2} \\ du = \sin 2x\, dx & dv = \sin^2 x\, dx \end{array}$$

and we obtain

$$\begin{aligned} \int \sin^4 x\, dx &= \sin^2 x \left(-\frac{\sin 2x}{4} + \frac{x}{2}\right) - \int (\sin 2x)\left(-\frac{\sin 2x}{4} + \frac{x}{2}\right)\, dx \\ &= \sin^2 x \left(-\frac{\sin 2x}{4} + \frac{x}{2}\right) + \int \frac{\sin^2 2x}{4}\, dx - \int x\frac{\sin 2x}{2}\, dx \end{aligned}$$

We know from part (a) that

$$\int \sin^2 2x\, dx = -\frac{\sin 4x}{4} + x$$

and to compute $\int x \sin 2x\, dx$ we choose

$$\begin{array}{ll} u = x & v = -\frac{1}{2}\cos 2x \\ du = 1\, dx & dv = \sin 2x\, dx \end{array}$$

and we obtain

$$\begin{aligned} \int x \sin 2x\, dx &= -\frac{1}{2} x \cos 2x + \int \cos 2x\, dx \\ &= -\frac{1}{2} x \cos 2x + \frac{1}{2}\sin 2x \end{aligned}$$

and so combining everything we obtain

$$\int \sin^4 x\, dx = \sin^2 x \left(-\frac{\sin 2x}{4} + \frac{x}{2}\right) - \frac{\sin 4x}{16} + \frac{x}{4} + \frac{1}{4} x \cos 2x - \frac{1}{4}\sin 2x$$

42. We choose

$$\begin{array}{ll} u = \cos^{n-1} x & v = \sin x \\ du = -(n-1)\cos^{n-2} x \sin x \, dx & dv = \cos x \, dx \end{array}$$

and we obtain

$$\begin{aligned} \int \cos^n x \, dx &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \sin^2 x \, dx \\ &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x (1 - \sin^2 x) \, dx \\ &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx \end{aligned}$$

So we have

$$n \int \cos^n x \, dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx$$

and we finally get

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

(b) From the formula, we get

$$\int \cos^2 x \, dx = \frac{1}{2} \cos x \sin x + \frac{1}{2} \int 1 \, dx = \frac{1}{2} \cos x \sin x + \frac{x}{2}$$

(c) From the formula, we get

$$\int \cos^4 x \, dx = \frac{1}{4} \cos^3 x \sin x + \frac{3}{4} \int \cos^2 x \, dx = \frac{1}{4} \cos^3 x \sin x + \frac{3}{4} \left(\frac{1}{2} \cos x \sin x + \frac{x}{2} \right)$$

45. If we let

$$\begin{array}{ll} u = x & v = \frac{(\ln x)^{n+1}}{n+1} \\ du = 1 \, dx & dv = \frac{(\ln x)^n}{x} \end{array}$$

then we obtain

$$\int (\ln x)^n \, dx = \frac{x (\ln x)^{n+1}}{n+1} - \frac{1}{n+1} \int (\ln x)^{n+1} \, dx$$

This re-arranges to

$$\int (lnx)^{n+1} \, dx = x (\ln x)^{n+1} - (n+1) \int (\ln x)^n \, dx$$

which is the same as what we want.

55. At some x-value, the shell is a cylinder with radius x and height $e^x - e^{-x}$. So the area is $2\pi x(e^x - e^{-x})$. Therefore, the integral we seek is

$$\int_0^1 2\pi x (e^x - e^{-x}) \, dx = 2\pi \left(\int_0^1 x e^x \, dx - \int_0^1 x e^{-x} \, dx \right)$$

For the first integral, if we let

$$\begin{array}{ll} u = x & v = e^x \\ du = 1 \, dx & dv = e^x \, dx \end{array}$$

Then we obtain

$$\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x$$

Likewise, for the second integral, if we let

$$\begin{array}{ll} u = x & v = -e^{-x} \\ du = 1\ dx & dv = e^{-x} \end{array}$$

and we obtain

$$\int xe^{-x}\ dx = -xe^{-x} + \int e^{-x}\ dx = -xe^{-x} - e^{-x}$$

So, back to what we want, we have

$$2\pi \left(\int_0^1 xe^x\ dx - \int_0^1 xe^{-x}\ dx \right) = 2\pi((xe^x - e^x)|_0^1 - (-xe^{-x} - e^{-x})|_0^1) = \frac{4\pi}{e}$$

62. If we let

$$\begin{array}{ll} u = f(x) & v = g'(x) \\ du = f'(x)\ dx & dv = g''(x)\ dx \end{array}$$

Then we have

$$\int f(x)g''(x)\ dx = f(x)g'(x) - \int f'(x)g'(x)\ dx$$

For our new integral, if we let

$$\begin{array}{ll} u = f'(x) & v = g(x) \\ du = f''(x)\ dx & dv = g'(x)\ dx \end{array}$$

then we get

$$\int f'(x)g'(x)\ dx = f'(x)g(x) - \int f''(x)g(x)\ dx$$

So overall, we obtain

$$\int f(x)g''(x)\ dx = f(x)g'(x) - \left(f'(x)g(x) - \int f''(x)g(x)\ dx \right) = f(x)g'(x) - f'(x)g(x) + \int f''(x)g(x)\ dx$$