Solutions: 7.3

7. We need to integrate

$$\int \frac{1}{x^2\sqrt{25-x^2}}\,dx$$

we make the substitution $x = 5\cos u$, which gives $dx = -5\sin u$ and we have

$$\begin{aligned}\int \frac{1}{x^2\sqrt{25-x^2}}\,dx &= \int \frac{-5\sin u\,du}{25\cos^2 u\sqrt{25-25\cos^2 u}} \\ &= \frac{-1}{25}\int \sec^2 u\,du \\ &= \frac{-1}{25}\tan u \\ &= \frac{-1}{25}\tan(\cos(\frac{x}{5}))\end{aligned}$$

and by drawing a triangle, we see that

$$\tan(\cos(\frac{x}{5})) = \frac{\sqrt{25-x^2}}{x}$$

so we have

$$\int \frac{1}{x^2\sqrt{25-x^2}}\,dx = \frac{-1}{25}\frac{\sqrt{25-x^2}}{x} + C$$

12. We need to integrate

$$\int x\sqrt{x^2+4}\,dx$$

so we make the substitution $x = 2\tan u$ so $dx = 2\sec^2 u\,du$ and we get

$$\begin{aligned}\int x\sqrt{x^2+4}\,dx &= \int 2\tan u\sqrt{4\tan^2 u+4}\,du \\ &= 4\int \tan^2 u\,du \\ &= 4(\tan u + u)\end{aligned}$$

Note that $u = \tan^{-1}\frac{x}{2}$ so we have

$$\int x\sqrt{x^2+4}\,dx = 4(\tan\tan^{-1}\frac{x}{2} + \tan^{-1}\frac{x}{2}) = 2x + 4\tan^{-1}\frac{x}{2} + C$$

so

$$\int_0^1 x\sqrt{x^2+4}\,dx = (2x+4\tan^{-1}\frac{x}{2})|_0^1 = (2+4\tan^{-1}\frac{1}{2}) - (0+\tan^{-1}0) = 2+4\tan^{-1}\frac{1}{2}$$

13. We need to integrate

$$\int \frac{x^2-9}{x^3}\,dx$$

so we make the substitution $x = 3\sec u$ so $dx = 3\tan u \sec u\, du$ so we obtain

$$\begin{aligned}
\int \frac{x^2-9}{x^3}\, dx &= \int \frac{(3\tan u)(3\tan u \sec u)\, du}{27\sec^3 u} \\
&= \frac{1}{3}\int \frac{\tan^2 u}{\sec^2 u}\, du \\
&= \frac{1}{3}\int \sin^2 u\, du \\
&= \frac{1}{3}\left(\frac{1}{2}\left(u - \frac{1}{2}\sin 2u\right)\right) \\
&= \frac{1}{6}\left(u - \sin u \cos u\right) \\
&= \frac{\sec^{-1}\frac{x}{3}}{6} - \frac{\sin\sec^{-1}\frac{x}{3}\cos\sec^{-1}\frac{x}{3}}{6}
\end{aligned}$$

by drawing the triangle we see that $\sin\sec^{-1}\frac{x}{3} = \frac{\sqrt{x^2-9}}{x}$ and $\cos\sec^{-1}\frac{x}{3} = \frac{3}{x}$. So we have

$$\int \frac{x^2-9}{x^3}\, dx = \frac{\sec^{-1}\frac{x}{3}}{6} - \frac{\sqrt{x^2-9}}{2x^2}$$

16. We need to calculate

$$\int \frac{dx}{x^2\sqrt{16x^2-9}}$$

and we make the substitution $x = \frac{3}{4}\sec u$ with $dx = \frac{3}{4}\tan u \sec u$ and we get

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{16x^2-9}} &= \int \frac{\frac{3}{4}\tan u \sec u\, du}{(\frac{3}{4})^2\sec^2 u \cdot 3\tan u} \\
&= \frac{4}{9}\int \cos u\, du \\
&= \frac{4}{9}\sin\sec^{-1}\frac{4}{3x}
\end{aligned}$$

and by drawing a triangle we see that $\sin\sec^{-1}\frac{4}{3x} = \frac{\sqrt{16x^2-9}}{4x}$ so

$$\int \frac{dx}{x^2\sqrt{16x^2-9}} = \frac{\sqrt{16x^2-9}}{9x}$$

20. We make the substitution $u = 25 - t^2$ so $du = -2t\, dt$ and so

$$\begin{aligned}
\int \frac{t}{\sqrt{25-t^2}}\, dt &= -\int \frac{1}{2\sqrt{u}}\, du \\
&= -\sqrt{u} \\
&= -\sqrt{25-t^2} + C
\end{aligned}$$