Solutions: 7.4

1. (a)

$$\frac{2x}{(x+3)(3x+1)} = \frac{A}{x+3} + \frac{B}{3x+1}$$

(b)

$$\frac{1}{x^3+2x^2+x} = \frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{Cx+D}{(x+1)^2}$$

3. (a)

$$\frac{2}{x^2+3x-4} = \frac{2}{(x+4)(x-1)} = \frac{A}{x+4} + \frac{B}{x-1}$$

(b)

$$\frac{x^2}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Cx+D}{x^2+x+1}$$

5. (a)

$$\frac{x^4}{x^4-1} = 1 + \frac{1}{x^4-1} = 1 + \frac{1}{(x^2+1)(x-1)(x+1)} = 1 + \frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{x-1}$$

(b)

$$\frac{t^4+t^2+1}{(t^2+1)(t^2+4)^2} = \frac{At+B}{t^2+1} + \frac{Ct+D}{t^2+4} + \frac{Et+F}{(t^2+4)^2}$$

9. We need to integrate

$$\int \frac{x-9}{(x+5)(x-2)}\,dx$$

We set up partial fractions:

$$\frac{x-9}{(x+5)(x-2)} = \frac{A}{x+5} + \frac{B}{x-2}$$

The solution is $A = -4/3$ and $B = 7/3$. So we have

$$\int \frac{x-9}{(x+5)(x-2)}\,dx = -\frac{4}{3}\int \frac{1}{x+5}\,dx + \frac{7}{3}\int \frac{1}{x-2}\,dx = -\frac{4}{3}\ln|x+5| + \frac{7}{3}\ln|x-2| + C$$

15. We do not need partial fractions:

$$\frac{2x+3}{(x+1)^2} = \frac{2(x+1)+1}{(x+1)^2} = \frac{2}{x+1} + \frac{1}{(x+1)^2}$$

so we get

$$\int \frac{2x+3}{(x+1)^2}\,dx = 2\int \frac{1}{x+1}\,dx + \int \frac{1}{(x+1)^2}\,dx = 2\ln|x+1| - \frac{1}{x+1} + C$$

16. Note that

$$\frac{x^3-4x-10}{x^2-x-6} = x+1+\frac{3x-4}{x^2-x-6}$$

so we set up the partial fraction

$$\frac{3x-4}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}$$

which gives the solution $A = 1$ and $B = 2$. So we get

$$\begin{aligned}\int \frac{x^3 - 4x - 10}{x^2 - x - 6}\, dx &= \int x\, dx + \int 1\, dx + \int \frac{1}{x-3}\, dx + 2\int \frac{1}{x+2}\, dx \\ &= \frac{x^2}{2} + x + \ln|x-3| + 2\ln|x+2| + C\end{aligned}$$

20. We set up the partial fraction

$$\frac{x^2}{(x-3)(x+2)^2} = \frac{A}{x-3} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

which yields the solutions

$$\begin{aligned}A &= \frac{9}{25} \\ B &= \frac{16}{25} \\ C &= -\frac{4}{5}\end{aligned}$$

So we get

$$\begin{aligned}\int \frac{x^2}{(x-3)(x+2)^2}\, dx &= \frac{9}{25}\int \frac{1}{x-3}\, dx + \frac{16}{25}\int \frac{1}{x+2}\, dx - \frac{4}{5}\int \frac{1}{(x+2)^2}\, dx \\ &= \frac{9}{25}\ln|x-3| + \frac{16}{25}\ln|x+2| + \frac{4}{5(x+2)} + C\end{aligned}$$

28. We set up the partial fraction

$$\frac{x^2 - 2x - 1}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx + D}{x^2+1}$$

which yields the solution

$$\begin{aligned}A &= 1 \\ B &= -1 \\ C &= -1 \\ D &= 1\end{aligned}$$

So we get

$$\begin{aligned}\int \frac{x^2 - 2x - 1}{(x-1)^2(x^2+1)}\, dx &= \int \frac{1}{x-1}\, dx - \int \frac{1}{(x-1)^2}\, dx + \int \frac{1}{x^2+1}\, dx - \int \frac{x}{x^2+1}\, dx \\ &= \ln|x-1| + \frac{1}{x-1} + \tan^{-1} x - \frac{\ln|x^2+1|}{2}\end{aligned}$$

45. Note that

$$\frac{1}{\sqrt{x} - \sqrt[3]{x}} = \frac{1}{x^{5/6}(x^{-1/3} - x^{-1/2})} = \frac{1}{x^{5/6}(\frac{1}{\sqrt[3]{x}} - \frac{1}{\sqrt{x}})}$$

So if we make the substitution $u = \sqrt[6]{x}$ then we get $du = \frac{1}{6}x^{-5/6}$ and so we get

$$\int \frac{1}{x^{5/6}(\frac{1}{\sqrt[3]{x}} - \frac{1}{\sqrt{x}})}\, dx = 6\int \frac{du}{\frac{1}{u^2} - \frac{1}{u^3}}\, du = 6\int \frac{u^3}{u-1}\, du$$

Now note that

$$\frac{u^3}{u-1} = u^2 + u + 1 + \frac{1}{u-1}$$

so we get

$$6\int \frac{u^3}{u-1}\,du = \int u^2 + u + 1\,du + \int \frac{1}{u+1}\,du = \frac{u^3}{3} + \frac{u^2}{2} + u + \ln|u+1| + C$$

and by substituting back we get

$$\int \frac{1}{\sqrt{x} - \sqrt[3]{x}} = \frac{\sqrt{x}}{3} + \frac{\sqrt[3]{x}}{2} + \sqrt[6]{x} + \ln|\sqrt[6]{x} + 1| + C$$

48. If we make the substitution $u = \sin x$, then $du = \cos x\,dx$ and we get

$$\int \frac{\cos x}{\sin^2 x + \sin x}\,dx = \int \frac{du}{u^2 + u}$$

We set up the partial fraction

$$\frac{1}{u^2 + u} = \frac{A}{u} + \frac{B}{u+1}$$

which yields the solution $A = 1$ and $B = -1$. Therefore,

$$\int \frac{du}{u^2 + u} = \int \frac{du}{u} - \int \frac{du}{u+1} = \ln|u| - \ln|u+1| + C$$

and substituting back we get

$$\int \frac{\cos x}{\sin^2 x + \sin x}\,dx = \ln|\sin x| - \ln|\sin x + 1| + C$$

60. We need to find

$$\int \frac{1}{x^2 - 6x + 8}\,dx$$

So we have the partial fraction decomposition

$$\frac{1}{x^2 - 6x + 8} = \frac{A}{x-4} + \frac{B}{x+2}$$

which yields the solution $A = \frac{1}{2}$ and $B = -\frac{1}{2}$. So we get

$$\int \frac{1}{x^2 - 6x + 8}\,dx = \frac{1}{2}\int \frac{1}{x-4}\,dx - \frac{1}{2}\int \frac{1}{x-2}\,dx = \frac{1}{2}\ln|x-4| - \frac{1}{2}\ln|x-2| + C$$