17 July 2006

Quiz 21 - Math 152

Approximate $f(x) = \frac{1}{x^2}$ with a Taylor Expansion around $x = 2$ with degree $n = 4$. How does this approximation compare to the actual value of $f(3)$?

Solution.

As with any Taylor Series problem, we start writing derivatives:

$$\begin{aligned} f(x) &= x^{-2} \\ f'(x) &= -2x^{-3} \\ f''(x) &= 6x^{-4} \\ f'''(x) &= -24x^{-5} \\ f''''(x) &= 120x^{-6} \end{aligned}$$

So notice that

$$f^{(n)}(x) = (-1)^n (n+1)! x^{-(n+2)}$$

and so we get that

$$c_n = \frac{f^{(n)}(2)}{n!} = \frac{(-1)^n (n+1)! 2^{-(n+2)}}{n!} = (-1)^n \frac{n+1}{2^{n+2}}$$

so we have

$$\begin{aligned} c_0 &= \frac{1}{4} \\ c_1 &= -\frac{1}{4} \\ c_2 &= \frac{3}{16} \\ c_3 &= -\frac{1}{8} \\ c_4 &= \frac{5}{64} \end{aligned}$$

And so we have

$$f(x) \approx \frac{1}{4} - \frac{1}{4}(x-2) + \frac{3}{16}(x-2)^2 - \frac{1}{8}(x-2)^3 + \frac{5}{64}(x-2)^4$$

The approximation at $x = 3$ is

$$f(x) \approx \frac{1}{4} - \frac{1}{4} + \frac{3}{16} - \frac{1}{8} + \frac{5}{64} = \frac{9}{64} = .140625$$

Whereas

$$f(3) = \frac{1}{9} \approx .111111$$