Workshop: Taylor and Maclaurin Series

Hello class,

Taylor and Maclaurin series are very, very nice tools to have at your disposal. I felt that having a workshop on it would be more beneficial than simply a lecture or two. There is a lot of computation involved, as the theory behind it isn't really that deep.

You will be in groups of either three or four. Please write down your names below. Depending on how good your answers are, up to 10 bonus quiz points will be added to your final grade. The quiz points will come after your three lowest grades are dropped for maximum effect.

Quality and clarity is the key. Explanations are crucial in this workshop. Use plenty of scratch paper, and place your final result, which should be in a clear and linear manner, on this handout.

Do this workshop in order. I know it is long, and it's really okay not to finish. I do hope, however, that you all get to at least the end of problem 4).

If you're stuck, then ask, ask, ask.

There will still be a quiz on Wednesday! I will be around to assist on this workshop. This should take the full two hours, with a 10- to 15-minute break in between. Good luck!

Name:

Name:

Name:

Name:

Warming up. Recall that we started out with the power series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad |x| < 1$$

We can do any sort of usual manipulation to these functions, including addition, multiplication, differentiation, integration, etc. With that in mind, find the power series representation for the following functions:

(a)

$$\frac{x}{(1-x)^3}$$

(b)

$$\frac{x^2-1}{1-x}$$

Now we will introduce *Taylor's Theorem*, which will basically find the power series representation for almost any function we want. The premise is simple: we have a function $f(x)$, and we want to express it as a power series centered at a, so it will look like

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

Our job is to find the coefficients c_i. Notice that

$$f(a) = \sum_{n=0}^{\infty} c_n (a-a)^n = c_0$$

So we have our first coefficient, namely $c_0 = f(a)$.

(a) Take the derivative of f and show that

$$c_1 = f'(a)$$

(b) Take the second derivative of f and show that

$$c_2 = \frac{f''(a)}{2}$$

(c) Continue this process to find c_3, c_4, and c_5.

(d) Based on your answers from (a)-(c), give a formula for an arbitrary coefficient c_n. Recall some notation: $f^{(n)}$ stands for the n^{th} derivative of f, and $n! = n(n-1)(n-2)(n-3)\cdots 3 \cdot 2 \cdot 1$.

2. Now we have a method to find power series representations, so let's see if it works. The question may have occurred to you recently:

Are power series representations *unique*?

We haven't dealt with this question, and frankly, we won't. Be assured, though, that these representations are unique. Taylor's Theorem gives us a way to get the coefficients, so we would hope that if we use Taylor's Theorem on a polynomial, we would get back the same polynomial.

So, show that for $f(x) = 3x^3 + x^2 - 4x + 4$, Taylor's Theorem gives us back the same polynomial. (Note: in how the polynomial is written, this is a power series centered at 0.)

(For the rest of your course and beyond, if a power series representation is asked for not specifying where it is centered around, then it is centered around zero).

3. Find the power series representations for $\sin x$ and $\cos x$.

4. (a) Notice that $\sin x$ is odd, and $\cos x$ is even. What is the power series representation for $\sin x + \cos x$?

(b) The answer to (a) looks really close to the power series representation for e^x - the only problem is that the signs go in the pattern $++--++--++--++--\cdots$.

Recall our third most-favorite number, i. i satisfies $i^2 = -1$. Using power series representations, show that

$$e^{ix} = \cos x + i \sin x$$

The formula $e^{ix} = \cos x + i \sin x$ is called *Euler's Formula*, after Leonhard Euler, one of the biggest mack-daddies of mathematics. It is an amazing formula, and it lays in the foundation of *complex analysis*, which is calculus with the Real Numbers with i added in there. One would think that just adding a teensy-weensy number like i wouldn't do much, but it does.

(a) Use Euler's Formula to find $e^{i\pi}$.

(b) Use Euler's Formula to find i^i. (Hint: first express i as e^{ix} for some x.)